Practical Diakoptics
for Electrical Networks

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Preface

The need to obtain numerical solutions of electrical networks can result from electrical circuit problems as such. It can also arise from equivalent circuits which are analogies of other physical situations. However, the greatest number of numerical solutions of networks probably occurs in connection with studies of electrical power systems. Accordingly the application of digital computers to these problems has received much attention.

The rewards for optimizing the design and the performance of electrical power systems increase with the size of the system but, unfortunately so does the severity of the problems involved in the analysis. Despite the improvements in the size and speed of digital computers used for this purpose it is always necessary to examine methods which lead to reduced storage requirements or to increased speeds or both. At the same time the preservation of accuracy is increasingly difficult. This is true for all analysis associated with system planning but particularly for the on-line applications of digital computers which are currently being considered for network control and scheduling. In these applications the analytical technique is a vital factor in selecting a suitable computer for the purpose.

The techniques described in this book were developed and applied in power system network analysis and many of the examples are taken from this field. However, as the title implies, the methods themselves are of general application to electrical network problems irrespective of their origin.

Diakoptics, from the Greek dia = through, and kopto = to tear, is the name given by Gabriel Kron to his method of subdividing large physical problems to enable their solution to be simplified. It is more than thirty years since Kron in his first papers on tensor analysis laid the foundations of the method of ‘tearing’ and since then many have studied and applauded Kron’s methods but few have applied them. The authors of this book believe that the problems arising in the application of these methods are essentially of a practical nature, being concerned with manipulative aspects of matrices. We further believe that to understand the problems involved it is essential to apply the methods to
given physical systems and to obtain valid solutions. Solutions in general symbolic form are not sufficient for this purpose. Such practical work appears not to have caught the imagination of most investigators and the subject has largely remained of academic interest only. Until the practical problems of their application are overcome, the outstanding advantages of diakoptic methods will not be appreciated. An added difficulty in following and interpreting Kron's work lies in his and his followers' use of a most general descriptive notation. This notation, involving the manipulation of subscripts and superscripts, is extremely powerful and indicates the common aspects of the method when applied to many different fields. However, when it is expressed in this notation even the simplest problem in a specific field can look awesome.

The purpose of this book is to explain to students and to practising engineers the techniques and advantages of diakoptics by applying the method in detail to some network problems arising from real physical situations. The approach here is to extend basic principles to cope with such problems rather than to present simplified examples in terms of general analysis. Treatment throughout the book is directed towards the use of methods most suitable for automatic digital computation.

Each chapter is as self-contained as possible and consists of theory and a general symbolic solution followed by a specific example. Throughout the book the theory is developed separately for mesh and nodal analyses so that appropriate equations are available for these complementary cases. The equations are developed in a simple matrix notation and the steps of the solution are summarized in tabulated form. These routine steps can be used directly as a basis for digital computer programs. Problems are also provided for the readers' exercise.

The origin of matrix algebra is closely associated with vector co-ordinate geometry. A vector can be considered as a quantity basically unchanging but which may be defined by different components in different co-ordinate systems. The relationship between the systems of description leads to definite relationships between the components of the vector described in each system. The corresponding linear transformations are often used to demonstrate the processes of matrix algebra. However, in this book the introduction and development of matrix algebra has been
related to electrical networks because their solution is its prime purpose.

The early chapters deal with terminology, matrix algebra, and network and tensor analysis together with a survey of standard numerical methods of solution. These subjects are taken only so far as is considered necessary to understand their application in the latter part of the book. Adequate references are provided for those wishing to read further. In chapter 4 the theory of transformation is developed in alternative ways using both invariance of power and topological concepts so that readers can choose the exposition which most appeals to them. The remaining chapters are concerned with applications, practical advice and some special cases. Finally, in chapter 8, the fundamental equations of diakoptics are extended to cover a variety of networks likely to be met in practice.

We have chosen to avoid the conflict surrounding the formal proof of Kron’s concepts because such a discussion would be out of place in this, a book of applications rather than a further exposition of the theory. His supporters and critics alike agree that the methods work. We have also avoided the use of some special terms such as intersection network since these are confusing. Kron’s orthogonal networks are here called mixed networks and we refer to branches removed in order to create subdivisions as removed branches rather than torn branches. Collectively the removed branches are called the removed network. We hope that these minor changes of terminology as compared to some published work will add to the consistency of this text and make it more easy to read.

The problems of notation caused us much concern. The rigid index notation of the generalized theory has been replaced here by normal matrix notation with subscripts used as identifiers. The problems of retaining consistent groups of identifiers between the simplest problems in the book and the most advanced were difficult. We feel we have achieved this desirable feature. All quantities are defined as they arise in each chapter but a summary of the important notation common to the later chapters is given at the end of this Preface.

It is appropriate for the authors to give their views here on the position of diakoptics relative to other methods for the analysis of physical systems. These views are based not only on the
present work and its extension to non-linear power system load-flow problems but also on applications in the fields of stress analysis and eigenvalue solutions of mechanical systems. We see diakoptics as a formal method for taking advantage of the physical characteristics of large problems in which the work of solution can be considerably reduced by judicious separation of interconnected areas. Diakoptics can be used with direct or iterative methods. In the former case it can increase the overall accuracy for a given working accuracy and in the latter case it can increase the convergence rate. For small problems the gain may not be worth while when the additional work involved in the method itself is considered. For larger problems the method is always worth while and for the largest problems its use may be essential to preserve accuracy or to cope with limited direct access digital storage.

It is hoped that the practical advice given in chapter 7 will encourage other workers to try out these techniques on real problems particularly those where existing techniques are inadequate. The true value of the method will then become apparent. On this basis interested readers may then find it possible to extend the techniques to other fields and may even be encouraged to study the more generalized theory.

The work described in the early chapters arose out of the lectures given by Brameller and John in the Advanced Engineering Course of Associated Electrical Industries Limited. Some of the remaining work has appeared in I.E.E. papers written while these authors were in the Power Systems Engineering Department of that Company. Most of the diakoptics material appeared in a thesis submitted by Brameller and approved for the award of the Degree of Ph.D. by the Council for National Academic Awards. The other authors were associated with this work as academic and industrial supervisors.

The authors acknowledge that this book is published with the permission of the Power Group of Associated Electrical Industries Limited and special acknowledgement is made to Dr. J.R. Mortlock, Technical Director of the Power Group, for his encouragement throughout the period of the work. It is recognized that suggestions from many colleagues have contributed to the book and special thanks are due to Mr. D.W. Mortifee for his editorial assistance. Finally, but by no means least, our thanks are
due to Mrs. W. Holt for the volumes of difficult typing involved in the preparation of a text of this nature.

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Manchester 1967
Principles of Common Notation in Chapters 5-8

**Base letters**  
$Z$ denotes a physical quantity or element  
$\mathbf{Z}$ denotes the corresponding matrix  
The use of small or capital base letters is reserved, thus  
$i$ denotes an unknown quantity – to be solved for  
$l$ denotes a given quantity or ‘driving function’  

**Subscripts**  
$\alpha$ denotes nodal quantities, $\alpha = a, b, c, \ldots$,  
$\beta$ denotes mesh quantities, $\beta = p, q, r, \ldots$,  
$\psi$ denotes removed branch quantities  
$\psi = j, k, l, \ldots$,  

**Identifier**  
$\sim$ The use of $\sim$ over a letter identifies a quantity arising from subdivision.  

**Examples**

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CHAPTER ONE

Matrices

This chapter presents briefly only the necessary amount of matrix theory on which the rest of the book is based. The reader who requires a more extended treatment of matrix theory may refer to a number of books on this topic [1, 2, 3].

Origin and notation

Consider the electrical network represented diagrammatically in fig. 1.1. In chapter 2 it will be shown that the following set of

\[
\begin{align*}
(R_1 + R_4)i_1 - R_4i_2 &= E_1 \\
-R_4i_1 + (R_2 + R_4 + R_5)i_2 - R_5i_3 &= 0 \\
-R_5i_2 + (R_3 + R_5)i_3 &= -E_3
\end{align*}
\] (1.1)

In these equations there are three sets of letters representing three different physical quantities which can be identified in the network. A letter which represents the same quantity is often called a base letter and is shown in italics. In the above example \(E\) has been used for e.m.f., \(R\) for resistance and \(i\) for mesh current. Each element in the equation (1.1) is made up of a base letter with a number attached to it which distinguishes it from other elements with the same base letter. The number is often called a subscript or an index. Throughout this book a small
letter is used to signify an unknown quantity and a capital letter represents a known quantity or a driving function.

It is quite natural to use subscripts, but it is sometimes necessary to use superscripts, thus equation (1.1) could also be written as

\[
\begin{align*}
(R^1 + R^4)i^1 - R^4 i^2 &= E^1 \\
-R^4 i^1 + (R^2 + R^4 + R^5)i^2 - R^5 i^3 &= 0 \\
-R^5 i^2 + (R^3 + R^5)i^3 &= -E^3
\end{align*}
\]

and would mean exactly the same. An application of superscripts is in connection with the use of complex numbers (see p. 21).

Matrix notation is simply a condensed form of writing a set of linear equations. All elements belonging to one quantity can be represented by arrays of numbers. For example, the quantities related by equation (1.1) can be specified by the three arrays of numbers

\[
\mathbf{R} = \begin{bmatrix}
(R_1 + R_4) & -R_4 & 0 \\
-R_4 & (R_2 + R_4 + R_5) & -R_5 \\
0 & -R_5 & (R_3 + R_5)
\end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} E_1 \\ 0 \\ -E_3 \end{bmatrix}
\]

The matrices \( \mathbf{E}, \mathbf{R} \) and \( \mathbf{i} \) are thus defined in terms of the physical or mesh quantities such as \( E_1, R_4, i_2 \).

There are a number of different types of brackets used for enclosing the arrays, for example ( ), { }, || or [ ]. The practice adopted in this book, in common with most engineering literature, is to enclose arrays in long square brackets and represent them by a single letter printed in sans serif bold type.

Sometimes it is more convenient if each element in a matrix is denoted by a general symbol. Considerable advantage is obtained if each element of a matrix is denoted by the same letter with a single or double subscript to distinguish it from the other elements. The resistance matrix \( \mathbf{R} \) could be written as

\[
\mathbf{R} = \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
\]
By convention, the first suffix indicates the row number and the second the column number. For example, $R_{21}$ is the element in the second row and first column. Numerically, $R_{21} = -R_4$, $R_{31} = 0$, $R_{33} = R_3 + R_5$ and so on in the matrix from equation (1.1).

When an array consists of a single column, one subscript only is used to indicate the row position. For example, in the matrix $i,$

$$
i = \begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix}
$$

$i_3$ indicates the element in the third row and numerically is equal to the mesh current $i_3$.

Sometimes it is useful to extend the subscript notation to the matrix symbols. For example, a matrix $R$ with $\alpha$ rows and $\beta$ columns is indicated by $R_{\alpha\beta}$.

By convention the first subscript indicates the number of rows and the second the number of columns. The subscripts will often be omitted in this book if there is no ambiguity.

Definition

A matrix is a rectangular array of elements arranged in rows and columns. The elements can be real or complex numbers or even matrices themselves as will be shown later. The order of a matrix is the number of rows and the number of columns. Thus the matrix $R_{\alpha\beta}$ is of the order $\alpha$ by $\beta$ and is quoted in this form.

Special matrices in common use

In mathematics, special types of matrices are frequently encountered. Some of these, which are used in this book, will now be defined.

Square matrix

A matrix with the same number of rows and columns is called a square matrix:

$$
R = \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
$$
In subscript notation a square matrix is shown as $R_{\beta \beta}$, $\beta$ is the order of the matrix.

The principal diagonal of a square matrix of order $\beta$ comprises the elements $R_{11}, R_{22}, \ldots, R_{\beta \beta}$.

**Column matrix**

A matrix consisting of an array of elements in a single column is called a column matrix. Because of the historical relationship with co-ordinate geometry, column matrices are known as column vectors:

$$
i = \begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix}
$$

In subscript notation a column matrix is shown as $i_\beta$, where $\beta$ indicates the number of rows and is called the order of the matrix.

**Row matrix**

A matrix consisting of an array of elements in a single row is called a row matrix or a row vector:

$$
E = [E_1 \ E_2 \ E_3]
$$

In subscript notation a row matrix is shown as $E_\beta$, where $\beta$ indicates the number of columns and is called the order of the matrix. Although they may be represented by the same symbols, row and column indices are identifiable by their context.

**Transposed matrix**

The transposed form of a matrix is one in which the rows and columns are interchanged and is denoted by a superscript $t$. For instance, if

$$
X = \begin{bmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{bmatrix}
$$
then
\[ X^t = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \]

In subscript notation the transpose of matrix \( X_{\alpha\beta} \) is shown as \( X^t_{\beta\alpha} \). The superscript t merely indicates that a transpose operation has been performed on a matrix. The subscripts indicate the number of rows and columns respectively in the transposed matrix. The transpose of \( X^t \) is \( X \) again.

**Symmetrical matrix**

A square matrix which is symmetrical with respect to the principal diagonal is called a symmetrical matrix:

\[ X = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \]

For symmetrical matrices, \( X = X^t \).

**Diagonal matrix**

A square matrix in which all the elements except those on the principal diagonal are zero is called a diagonal matrix:

\[ X = \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{bmatrix} \]

**Unit matrix**

A square matrix which has unit elements on the principal diagonal and zero elements elsewhere is called a unit matrix:

\[ U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

The special symbol \( U \) will always be associated with a unit matrix.
Null matrix

A square or rectangular matrix with all elements equal to zero is called a null matrix and is denoted by \( \mathbf{0} \):

\[
\mathbf{0} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Matrix algebra

The systematic manipulation of arrays of coefficients and variables to achieve the required solutions are gathered together under the heading of matrix algebra.

The most important rules for the purpose of this book are summarized below.

Equality

Two matrices are equal only when they are of the same order and every element in one is equal to the corresponding element in the other.

Addition

The sum of two matrices is obtained by adding corresponding elements in each matrix. Thus the sum of matrices \( \mathbf{X}_{\alpha\beta} \) and \( \mathbf{Y}_{\alpha\beta} \), each with \( \alpha \) rows and \( \beta \) columns is a matrix also with \( \alpha \) rows and \( \beta \) columns. In general,

\[
\mathbf{X}_{\alpha\beta} + \mathbf{Y}_{\alpha\beta} = \mathbf{Z}_{\alpha\beta}
\]

where \( \mathbf{X}_{rc} + \mathbf{Y}_{rc} = \mathbf{Z}_{rc} \)

or in more detail,

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} +
\begin{bmatrix}
j & k & l \\
m & n & o \\
p & q & r
\end{bmatrix} =
\begin{bmatrix}
(a + j) & (b + k) & (c + l) \\
(d + m) & (e + n) & (f + o) \\
(g + p) & (h + q) & (i + r)
\end{bmatrix}
\]

Subtraction is treated in the same way.

For addition of two matrices to be possible it is necessary for both matrices to be of the same order.

The law of addition has two important properties in which matrix algebra is the same as ordinary algebra.
Matrices

The commutative property,
\[ X + Y = Y + X \]
and the associative property,
\[(X + Y) + Z = X + (Y + Z)\]

Addition of a null matrix to a matrix leaves the matrix unchanged
\[ R + O = R \]

Scalar multiplication
The summation rule may be easily extended to multiplication of a matrix by an integer. It can be seen that the matrix \( Z = X + X \) has element \( Z_{rc} = 2X_{rc} \) and \( Z = X + X + X \) has elements \( Z_{rc} = 3X_{rc} \).

It follows from the above that any matrix \( X \) can be multiplied by a scalar \( k \), where \( k \) is an integer, if each element of the matrix is multiplied by the given scalar number. In general,
\[ Z = kX = Xk \]
where \( Z_{rc} = kX_{rc} = X_{rc}k \)
thus
\[
\begin{bmatrix}
3 & 0 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
6 & 0 \\
-2 & 4
\end{bmatrix}
\]
In general, \( k \) can be any real or complex number.

Matrix multiplication
The matrix product can be explained by considering the following set of equations:
\[
R_{11}i_1 + R_{12}i_2 + R_{13}i_3 = E_1 \\
R_{21}i_1 + R_{22}i_2 + R_{23}i_3 = E_2 \\
R_{31}i_1 + R_{32}i_2 + R_{33}i_3 = E_3
\]
These equations may be considered as multiplication of the currents \( i \) by the various resistances \( R \) to give the voltages \( E \). In general matrix notation the equations can be represented by
\[ Ri = E \]
which in detailed matrix form is

\[
\begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix}
=
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}
\]

Matrix multiplication is so defined that the product of these two matrices \( R i \) yields the matrix \( E \). In general if \( R_{\alpha\beta} \) is a matrix with \( \alpha \) rows and \( \beta \) columns and \( i_{\beta\gamma} \) is a matrix with \( \beta \) rows and \( \gamma \) columns, the product \( R_{\alpha\beta} i_{\beta\gamma} \) is a matrix \( E_{\alpha\gamma} \) with \( \alpha \) rows and \( \gamma \) columns.

\[
R_{\alpha\beta} i_{\beta\gamma} = E_{\alpha\gamma}
\]

The elements of \( E_{\alpha\gamma} \) are given by a ‘row into column’ rule, i.e.

\[
E_{rc} = \sum_{k=1}^{\beta} R_{rk} i_{kc}
\]

The element \( E_{rc} \) in row \( r \) and column \( c \) is obtained by multiplying each element of \( R \) in row \( r \) by the corresponding element of \( i \) in column \( c \) and summing the product. For example,

\[
\begin{bmatrix}
a & b & c \\
d & e & f
\end{bmatrix}
\begin{bmatrix}
g & i \\
h & k
\end{bmatrix}
=
\begin{bmatrix}
(ag + bh + ci) & (aj + bk + cl) \\
(dg + eh + fi) & (dj + ek + fl)
\end{bmatrix}
\]

It follows that two matrices may be multiplied together only if the number of columns in the first is equal to the number of rows in the second, i.e.

\[
R_{\alpha\beta} i_{\beta\gamma} = E_{\alpha\gamma}
\]

It also follows from this that in matrix products neighbouring subscripts of different base symbols must be identical.

In this product \( R \) is said to pre-multiply \( i \) and \( i \) to post-multiply \( R \).

Matrix multiplication is non-commutative, i.e. except in special cases

\[
R i \neq i R
\]

One special case is the unit matrix \( U \), pre- or post-multiplication by which leaves a matrix unchanged. This may be verified
by performing the multiplications

\[
UR = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix} = R
\]

\[
RU = \begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = R
\]

Any number of matrices may be multiplied in sequence provided that the numbers of columns and rows are consistent in each product. The order of the resulting matrix is determined by the number of rows in the first and the number of columns in the last of the matrices to be multiplied together, i.e.

\[
A_{\alpha\beta}B_{\beta\gamma}C_{\gamma\delta}D_{\delta\theta} = E_{\alpha\theta}
\]

It may be demonstrated that the transpose of a product is obtained by transposing each matrix and taking the product in the reverse order, thus

\[
(A_{\alpha\beta}B_{\beta\gamma})^t = (B_{\beta\gamma})^t(A_{\alpha\beta})^t = B_{\gamma\beta}^tA_{\beta\alpha}^t
\]

E.g.

\[
AB = \begin{bmatrix}
a & b & c \\
d & e & f
\end{bmatrix}
\begin{bmatrix}
g & i \\
h & k \\
i & l
\end{bmatrix} = \begin{bmatrix}
(ag + bh + ci) & (aj + bk + cl) \\
(dg + eh + fi) & (dj + ek + fl)
\end{bmatrix}
\]

and

\[
B^tA^t = \begin{bmatrix}
g & h & i \\
j & k & l
\end{bmatrix}
\begin{bmatrix}
a & d \\
b & e \\
c & f
\end{bmatrix} = \begin{bmatrix}
(ga + hb + ic) & (gd + he + if) \\
(ja + kb + lc) & (jd + ke + lf)
\end{bmatrix}
\]

hence \(B^tA^t = (AB)^t\).

Cancellation

Sometimes equations of the following form are encountered

\[
A_{\alpha\beta}x_\beta = B_{\alpha\beta}x_\beta
\]
where $A_{a\beta}, B_{a\beta}$ are matrices and $x_\beta$ is a vector. It is important to establish in what circumstances the vector $x_\beta$ can be cancelled to yield the result

$$A_{a\beta} = B_{a\beta}$$

Thus, although

$$\begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

it is not true that

$$\begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 7 & 3 \end{bmatrix}$$

However, if the equation $A_{a\beta} x_\beta = B_{a\beta} x_\beta$ is true for any values of the elements of the vector $x_\beta$ then it must follow that

$$A_{a\beta} = B_{a\beta}$$

This can be shown as follows choosing for simplicity a two by two matrix.

If

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and choosing $x_1 = 1$ and $x_2 = 0$, then on multiplying out

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

that is

$$a_{11} = b_{11}$$

$$a_{21} = b_{21}$$

Similarly by choosing $x_1 = 0$ and $x_2 = 1$, gives

$$a_{12} = b_{12}$$

$$a_{22} = b_{22}$$

hence if $A_{a\beta} x_\beta = B_{a\beta} x_\beta$ for all $x_\beta$

then

$$A_{a\beta} = B_{a\beta}$$
Summary of the properties of multiplication

The law of multiplication has six important properties. These are:

(1) The product of two matrices is defined only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

(2) In general the sequence of matrix multiplications cannot be interchanged, that is matrix multiplication is non-commutative,

\[ XY \neq YX \]

In the algebra of real and complex numbers multiplication is commutative.

(3) When a number of matrices are to be multiplied, the multiplication can be performed in any succession. For example,

\[ XYZ = X(YZ) = (XY)Z \]

or

\[ x_{a\beta} y_{\beta\gamma} z_{\gamma\delta} = x_{a\beta} (yz)_{\beta\delta} = (xy)_{a\gamma} z_{\gamma\delta} = (xyz)_{a\delta} \]

(4) The usual rules for multiplication of bracketed quantities apply. For example,

\[ X(Y + Z) = XY + XZ \]

\[ (Y + Z)X = YX + ZX \]

(5) The transpose of a product of matrices is found by taking the product of the transpose of each matrix in the reverse order. Thus

\[ (XYZ)^t = Z^t Y^t X^t \]

(6) Multiplication by a null matrix yields a null matrix:

\[ R \times O = O \]

and \[ O \times R = O \]

Partitioning

It is sometimes convenient, because of the size of a matrix, to divide it into a number of matrices of lower order, called sub-matrices, which can be more easily manipulated. Sometimes such a partitioning is desirable in order to separate one group of elements from another. The division is carried out by drawing horizontal and vertical lines through the matrix. As an example,
consider the partitioning of the matrix $X$ into four parts in the following manner

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}$$

This can be written in the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where

$$X_{11} = \begin{bmatrix} X_{11} \end{bmatrix} \quad X_{12} = \begin{bmatrix} X_{12} & X_{13} \end{bmatrix}$$

$$X_{21} = \begin{bmatrix} X_{21} \\ X_{31} \end{bmatrix} \quad X_{22} = \begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix}$$

The precise way in which the partitioning is carried out is quite arbitrary, as long as the submatrices can be added or multiplied as the context requires, i.e. having consistent numbers of rows and columns. Matrices in which each element is itself a matrix are called compound matrices.

The rules for addition and multiplication are similar to those for matrices, except that the elements are submatrices. For example,

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} (X_{11} + Y_{11}) & (X_{12} + Y_{12}) \\ (X_{21} + Y_{21}) & (X_{22} + Y_{22}) \end{bmatrix}$$

and

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} (X_{11} Y_{11} + X_{12} Y_{21}) & (X_{11} Y_{12} + X_{12} Y_{22}) \\ (X_{21} Y_{11} + X_{22} Y_{21}) & (X_{21} Y_{12} + X_{22} Y_{22}) \end{bmatrix}$$

In taking a transpose of a partitioned matrix, the transpose of each element is also taken. Thus if
\[ X = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad \text{then} \quad X^t = \begin{bmatrix} P^t & R^t \\ Q^t & S^t \end{bmatrix} \]

The inverse matrix

In order to solve for the unknown \( i \), in the equation

\[ R i = E \]

it is necessary to multiply both sides by the reciprocal \( 1/R \)

thus

\[ \frac{1}{R} R i = \frac{1}{R} E \]

so

\[ i = \frac{E}{R} \]

In order to solve the set of equations represented by the matrix equation

\[ R i = E \]

it is necessary to find a matrix \( G \) which will 'annihilate' the matrix \( R \), in the same way that the reciprocal \( 1/R \) annihilates the factor \( R \) in the equation

\[ R i = E \]

This can be achieved if a matrix \( G \) exists such that

\[ G R = U \]

For, pre-multiplying both sides of the original equation by \( G \) yields

\[ G R i = GE \]

\[ U i = GE \]

\[ i = GE \]

It can be shown that if \( R \) is a square matrix, then in general the matrix \( G \) defined by \( G R = U \) can be found and is square and unique. This matrix is termed the inverse of \( R \) and is denoted by \( R^{-1} \). The multiplication of a matrix by its inverse is commutative, i.e.

\[ RR^{-1} = R^{-1} R = U \]

The process of calculating the inverse is called inversion but
some matrices although square may not be inverted for reasons to be seen in the next section. Such matrices are said to be singular. A matrix is said to be non-singular if it possesses an inverse.

Once the inverse has been obtained, solutions for any number of 'right-hand sides' or 'driving functions' may be obtained by simple processes of multiplication. Thus for various sets of impressed voltages $E^{(1)}$, $E^{(2)}$, $E^{(3)}$ the corresponding sets of network currents $i^{(1)}$, $i^{(2)}$, $i^{(3)}$ are given by

$$i^{(1)} = R^{-1}E^{(1)}$$

$$i^{(2)} = R^{-1}E^{(2)}$$

$$i^{(3)} = R^{-1}E^{(3)}$$

**Inversion**

Calculating the inverse of a matrix corresponds to solving a set of simultaneous equations. The processes involved are most easily demonstrated by considering the equation

$$Ri = E$$

and the solutions for the unknown currents $i$ when $E$ is any one column of a unit matrix. For a third-order system, for example,

$$R_{11}i_1 + R_{12}i_2 + R_{13}i_3 = 0 \quad 0 \quad 0$$

$$R_{21}i_1 + R_{22}i_2 + R_{23}i_3 = 0 \quad \text{or} \quad 1 \quad \text{or} \quad 0$$

$$R_{31}i_1 + R_{32}i_2 + R_{33}i_3 = 0 \quad 0 \quad 1$$

Denoting each different column $E$ as $E^{(1)}$, $E^{(2)}$, $E^{(3)}$, and the corresponding solutions as $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, then

$$R \begin{bmatrix} i_1^{(1)} \\ i_2^{(1)} \\ i_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad R \begin{bmatrix} i_1^{(2)} \\ i_2^{(2)} \\ i_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad R \begin{bmatrix} i_1^{(3)} \\ i_2^{(3)} \\ i_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or as

$$R i^{(1)} = E^{(1)} \quad R i^{(2)} = E^{(2)} \quad R i^{(3)} = E^{(3)}$$

which can be stated in a matrix partitioned form as
Matrices

\[
\begin{bmatrix}
1^{(1)} & 1^{(2)} & 1^{(3)}
\end{bmatrix}
\begin{bmatrix}
R
\end{bmatrix}
\begin{bmatrix}
E^{(1)}
E^{(2)}
E^{(3)}
\end{bmatrix}
\]

or as

\[
\begin{bmatrix}
\begin{bmatrix}
1^{(1)} & 1^{(2)} & 1^{(3)}
\end{bmatrix}
\begin{bmatrix}
2^{(1)} & 2^{(2)} & 2^{(3)}
\end{bmatrix}
\begin{bmatrix}
3^{(1)} & 3^{(2)} & 3^{(3)}
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
R
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{bmatrix}
\]

Comparing this with the definition of the inverse matrix it is seen that the matrix formed from the three vectors \(1^{(1)}, 1^{(2)}, 1^{(3)}\) is \(R^{-1}\), the inverse of the coefficient matrix \(R\).

Inversion of second-order matrices

The application of this method in the case of a second-order matrix can be illustrated by considering the following two simultaneous equations

\[\begin{align*}
R_{11} i_1 + R_{12} i_2 &= E_1 \\
R_{21} i_1 + R_{22} i_2 &= E_2
\end{align*}\]

By elimination of either variable the general solution can be expressed as

\[\begin{align*}
i_1 &= \frac{R_{22} E_1 - R_{12} E_2}{\Delta} \\
i_2 &= \frac{R_{11} E_2 - R_{21} E_1}{\Delta}
\end{align*}\]

where \(\Delta = R_{11} R_{22} - R_{12} R_{21}\)

Now inserting the elements of the second-order unit matrix results in

\[\begin{align*}
i_1^{(1)} &= \frac{R_{22}}{\Delta}, & i_1^{(2)} &= \frac{-R_{12}}{\Delta} \\
i_2^{(1)} &= \frac{-R_{21}}{\Delta}, & i_2^{(2)} &= \frac{R_{11}}{\Delta}
\end{align*}\]

Hence the inverse of a second-order matrix is
\[
R^{-1} = \frac{1}{\Delta} \begin{bmatrix}
R_{22} & -R_{12} \\
-R_{21} & R_{11}
\end{bmatrix}
\]

If \( \Delta = 0 \) then the inverse does not exist, i.e. the matrix is singular. This, of course, would result from an inconsistent set of equations in which the coefficients were related by the above equation defining \( \Delta \).

**Determinants**

The quantity \( \Delta \) derived above is known as the determinant of the matrix of the coefficients. It is denoted by enclosing the array of elements within vertical lines and is obtained by a process of cross-multiplication with allocated signs, thus

\[
\begin{vmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{vmatrix}
\]

\[
\Delta = |R| = R_{11} R_{22} - R_{12} R_{21}
\]

Determinants are convenient ways of handling the many terms involved in the solution of higher-order systems. The study of the properties of determinants is well covered in many books such as [2, 3], and only a few more facts are appropriate to the present work.

For each element of a determinant there is a quantity called its *minor* which is the value of the determinant left after eliminating the row and column containing the element. For example, the minor \( M_{12} \) of the element \( R_{12} \) in the following third-order determinant

\[
\begin{vmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{vmatrix}
\]

is
\[
M_{12} = \begin{vmatrix}
R_{21} & R_{23} \\
R_{31} & R_{33}
\end{vmatrix} = R_{21} R_{33} - R_{31} R_{23}
\]
An important property of determinants is that they may be expanded as the sum of the product of each of the elements in one row (or column) and its minor with an allocated sign. This provides a means for expanding high-order determinants in terms of lower-order ones. For example, a third-order determinant may be expanded thus from the first row.

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
= a \begin{vmatrix}
  e & f \\
  h & i \\
\end{vmatrix}
- b \begin{vmatrix}
  d & f \\
  g & i \\
\end{vmatrix}
+ c \begin{vmatrix}
  d & e \\
  g & h \\
\end{vmatrix}
\]

\[= a(ei - fh) - b(di - fg) + c(di - eg)\]

The signs to be used with the minors are indicated in the following diagram for systems of any order and the term co-factor is given to the signed minor of an element

\[
\begin{vmatrix}
  + & - & + & . & . \\
  - & + & . & . \\
  + & - & + & . & . \\
  . & . & . & + & - \\
  . & . & . & - & + \\
\end{vmatrix}
\]

**Inversion of higher-order matrices**

It may be shown that the inverse $\mathbf{R}^{-1}$ of a matrix $\mathbf{R}$ is obtained by applying the following rules:

1. Evaluate $|\mathbf{R}|$, the determinant of the matrix, by expanding into determinants of lower and lower order until second-order determinants can be calculated. The work can be reduced if rows or columns with the greatest number of zero elements are chosen for the expansions.

2. Replace each element $R_{rc}$ by its minor $M_{rc}$. The minors of one row or column will be available as the first levels in the expansion of the determinant.

3. Allocate signs to each minor, according to the general rule illustrated above for determinants, to obtain the co-factors.

4. Transpose the matrix of the co-factors; this is called the adjoint matrix.
(5) Divide each element by the determinant $\Delta (= |R|)$.

$$R^{-1} = \frac{1}{|R|} \begin{bmatrix}
M_{11} & -M_{21} & M_{31} \\
-M_{11} & M_{22} & -M_{32} \\
M_{13} & -M_{23} & M_{33}
\end{bmatrix}$$

It will be appreciated that a large number of numerical operations are involved and this is not a practical method for matrices of order higher than three. There are many ways of inverting matrices and of solving large systems of equations [4, 5]. Some of the practical methods for network equations are considered in chapter 3.

**Numerical examples of matrix inversion**

Consider the inverse of the matrix

$$R = \begin{bmatrix}
3 & 2 & 1 \\
1 & 2 & 1 \\
2 & 1 & 3
\end{bmatrix}$$

(1) The determinant, expanding from the first row, is

$$|R| = 3(2 \times 3 - 1 \times 1) - 2(1 \times 3 - 2 \times 1) + 1(1 \times 1 - 2 \times 2) = 10$$

(2) The matrix of minors is

$$\begin{bmatrix}
5 & 1 & -3 \\
5 & 7 & -1 \\
0 & 2 & 4
\end{bmatrix}$$

(3) and (4) Changing the sign of every alternate element and transposing gives

$$\begin{bmatrix}
5 & -5 & 0 \\
-1 & 7 & -2 \\
-3 & 1 & 4
\end{bmatrix}$$
(5) Finally dividing each element by the determinant $|R|$ gives the inverse

$$R^{-1} = \begin{bmatrix}
0.5 & -0.5 & 0 \\
-0.1 & 0.7 & -0.2 \\
-0.3 & 0.1 & 0.4
\end{bmatrix}$$

A check on the correctness of the inverse calculations can be obtained from

$$R^{-1}R = U$$

Some properties of the inverse matrix

(1) Only a square matrix has an inverse and the inverse matrix is square.

(2) The product of a matrix and its inverse is a unit matrix.

$$RR^{-1} = R^{-1}R = U$$

(3) If the determinant $|R|$ of the matrix $R$ is zero, the inverse $R^{-1}$ does not exist and $R$ is singular.

(4) If $R$ is a symmetrical matrix then $R^{-1}$ is also symmetrical.

(5) If $R$ is a diagonal matrix then $R^{-1}$ is also diagonal and is obtained by taking the reciprocal of each element $R$

$$\begin{bmatrix}
X & . & . \\
. & Y & . \\
. & . & Z
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{1}{X} & . & . \\
. & \frac{1}{Y} & . \\
. & . & \frac{1}{Z}
\end{bmatrix}$$

(6) If $R$ is a block diagonal matrix, i.e. a partitioned matrix such that all submatrices are zero except those on the principal diagonal, then $R^{-1}$ is also a block diagonal matrix and is obtained by taking the inverse of each individual submatrix, e.g.
$$\begin{bmatrix} X & . & . \\ . & Y & . \\ . & . & Z \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} & . & . \\ . & Y^{-1} & . \\ . & . & Z^{-1} \end{bmatrix}$$

This is a very important property in the application of diakoptics.

(7) The inverse of a matrix product is the product of the inverse of the separate matrices taken in the reverse order

$$(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}$$

It is suggested that the reader should demonstrate properties (4) to (7) to his own satisfaction by working with third-order matrices of his own construction.

Complex matrices

When calculating steady-state conditions in a.c. circuits, the values of the elements of networks are expressed by complex numbers. Matrices whose elements are complex numbers are called complex matrices. This does not affect the method of matrix algebra, but it does increase the number of calculations involved. Hand calculations with complex numbers are liable to errors and, when special digital computer procedures dealing with complex numbers are not available, it is sometimes more convenient to express the problem by an equation with real numbers only.

A matrix whose elements are complex numbers can be separated into two matrices, one consisting of real numbers and the other of imaginary numbers. For example, if $Z$ is a matrix with complex numbers having real parts $R$ and imaginary parts $X$, it may be written as

$$Z = R + jX$$

where $j = \sqrt{-1}$ the complex operator is a common factor of $X$.

Consider now the set of simultaneous linear equations with complex coefficients and variables written in matrix form as

$$Zi = E$$
where
\[ Z = R + jX \]
\[ i = i^p + ji^q \]
\[ E = E^p + jE^q \]

The superscripts \( p \) and \( q \) denote the in-phase or real parts and the quadrature or imaginary parts respectively of the quantities concerned. The set of simultaneous equations with complex numbers can be written with the real parts separated from the imaginary parts as
\[
(R + jX)(i^p + ji^q) = E^p + jE^q
\]

Multiplying out and equating the real parts and the imaginary parts results in
\[
R i^p - X i^q = E^p
\]
\[
X i^p + R i^q = E^q.
\]

Writing the last two equations in a compound matrix form,
\[
\begin{bmatrix}
R & -X \\
X & R
\end{bmatrix}
\begin{bmatrix}
i^p \\
i^q
\end{bmatrix}
= \begin{bmatrix}
E^p \\
E^q
\end{bmatrix}
\]

The original set of equations with complex numbers is now expressed by an equation with real numbers only and can be manipulated in the usual way.

As an example consider the following equations
\[
(1 + j2)(i^p_1 + ji^q_1) - 2(i^p_2 + ji^q_2) = 4 + j3
\]
\[
- j3(i^p_1 + ji^q_1) + (4 + j)(i^p_2 + ji^q_2) = 7 - j2
\]

In matrix form the set of equations can be written as
\[
\begin{bmatrix}
(1 + j2) & (-2 + j0) \\
(0 - j3) & (4 + j1)
\end{bmatrix}
\begin{bmatrix}
i^p_1 + ji^q_1 \\
i^p_2 + ji^q_2
\end{bmatrix}
= \begin{bmatrix}
4 + j3 \\
7 - j2
\end{bmatrix}
\]

Separating the real numbers from the imaginary numbers, the same set of equations can be written as
\[
\left( \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} + j \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \right) \left( \begin{bmatrix} i^p_1 \\ i^p_2 \end{bmatrix} + j \begin{bmatrix} i^q_1 \\ i^q_2 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 7 \end{bmatrix} + j \begin{bmatrix} 3 \\ -2 \end{bmatrix}
\]
The real equation can be written by picking out the various matrices by inspection, as

\[
\begin{bmatrix}
1 & -2 & -2 & 0 \\
0 & 4 & 3 & -1 \\
2 & 0 & 1 & -2 \\
-3 & 1 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
i_p^1 \\
i_p^2 \\
i_q^1 \\
i_q^2
\end{bmatrix}
=
\begin{bmatrix}
4 \\
7 \\
3 \\
-2
\end{bmatrix}
\]

**Inversion of a complex matrix**

The simplification of expressing a problem with complex numbers by means of real numbers only has been obtained at the expense of a larger matrix. The number of operations involved in inverting the real number matrices is doubled, as compared to inversion of the original matrix using complex arithmetic. The large number of calculations, however, can be reduced from the following consideration.

By definition, the product of a matrix and its inverse is a unit matrix. Therefore,

\[ZZ^{-1} = U\]

where \(Z\) is a square matrix with complex numbers,

\[Z = R + jX\]

and \(Z^{-1}\) is also a square matrix with complex numbers whose components can be defined as

\[Z^{-1} = G + jB\]

and \(U\) is a unit matrix.

Substituting for \(Z\) and \(Z^{-1}\)

\[(R + jX)(G + jB) = U + jO\]

Multiplying out and equating the real parts and imaginary parts

\[RG - XB = U\]  \hspace{1cm} (1.2)

\[XG + RB = O\]  \hspace{1cm} (1.3)

From equation (1.3)

\[B = -R^{-1}XG\]  \hspace{1cm} (1.4)
Substituting for \( B \) from equation (1.4) into equation (1.2)

\[
RG + XR^{-1}XG = U
\]

\[
\therefore (R + XR^{-1}X)G = U
\]

\[
\therefore G = (R + XR^{-1}X)^{-1}U = (R + XR^{-1}X)^{-1} \quad (1.5)
\]

Substituting for \( G \) from equation (1.5) into equation (1.4)

\[
B = -R^{-1}X(R + XR^{-1}X)^{-1} \quad (1.6)
\]

The inverse of \( Z \) has been defined as

\[
Z^{-1} = (R + jX)^{-1} = G + jB
\]

Substituting for \( G \) and \( B \) from equation (1.5) and (1.6)

\[
Z^{-1} = (R + jX)^{-1} = (R + XR^{-1}X)^{-1} - jR^{-1}X(R + XR^{-1}X)^{-1}
\]

Simplifying

\[
Z^{-1} = (R + jX)^{-1} = (U - jR^{-1}X)(R + XR^{-1}X)^{-1}
\]

The process of inverting a square matrix with complex numbers has now been reduced to the inversion of two matrices of the same order but with real numbers only, namely \( R \) and \( (R + XR^{-1}X) \). This not only simplifies the calculations but also reduces the amount of work considerably. The method described requires a certain number of matrix multiplications and additions and the necessity of preserving intermediate results.

Where special digital computer procedures dealing with complex arithmetic are available, the process of inverting a matrix with complex numbers can be further simplified. Some of the practical methods are described in detail in chapter 3.

Problems

1. Evaluate the determinants

\[
\begin{align*}
\text{a)} \quad \Delta_a &= \begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} & \text{b)} \quad \Delta_b &= \begin{vmatrix} 14 & 13 & 7 \\ 8 & 7 & 4 \\ 5 & 6 & 3 \end{vmatrix}
\end{align*}
\]
(c) $\Delta_c = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

(d) $\Delta_d = \begin{vmatrix} 3 & 3 & 4 & 1 \\ 2 & -3 & -4 & 2 \\ -5 & -2 & -2 & -1 \\ 4 & -4 & 2 & -1 \end{vmatrix}$

2. If $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 9 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & -3 \\ 9 & 4 & 2 \end{bmatrix}$

evaluate $C = AB$ and $D = BA$

3. Solve the following equations for the alternative right-hand sides

\[
x + y + z = 2 \quad 6 \\
x + 2y + 3z = 1 \quad \text{or} \quad 14 \\
3x + y - 5z = 4 \quad -10
\]

4. A set of three-phase currents $I$ is given in terms of three symmetrical component currents $i$ by the following equation

\[
\begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \lambda^2 & \lambda \\ 1 & \lambda & \lambda^2 \end{bmatrix} \begin{bmatrix} i_a \\ i_1 \\ i_2 \end{bmatrix}
\]

where $\lambda = e^{j2\pi/3}$

Determine, in similar form, the equations giving the component currents in terms of the phase values.

5. Solve the equations

\[(3 + j5)i_1 - (2 + j3)i_2 = 6\]
\[-(2 + j3)i_1 + (4 + j7)i_2 = 3 + j4\]

6. By suitable partitioning find the inverse of the matrix

\[
A = \begin{bmatrix} 3 & 4 & -2 & 6 & -1 \\ 0 & 2 & 0 & 7 & 0 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 3 & -1 & 0 & 2 \end{bmatrix}
\]
CHAPTER TWO
Network Analysis

The basic equations which can describe a given network are derived from first principles using only Ohm’s law and Kirchhoff’s laws. The object of using matrix algebra in network analysis is to establish and solve the equations in an organized manner. Experience has shown that difficulties encountered in the solution of some problems can be overcome by formulating the equations in a different way and the work involved in manipulating sets of variables for this purpose is considerably reduced by matrix methods.

In this chapter the standard methods of forming the equations are examined in detail. In later chapters these fundamental equations will be utilized in different ways to improve further the ease of formulation and solution, thus reducing the number of arithmetical operations required and hence the computation time involved. However, the following approach is considered desirable for an adequate understanding of the problem.

Network elements
Electrical systems can be reduced to equivalent networks which are composed of two kinds of elements, passive elements and active elements. A network is an assembly of interconnected passive and active elements.

Passive elements
These are elements which either absorb or store energy. In this book three direct types of linear passive elements are considered; resistors, $R$, inductors, $L$, and capacitors, $C$, which are symbolically represented as shown in fig. 2.1(a), 2.1(b), and 2.1(c). In addition, mutual inductance, corresponding to the indirect or electromagnetic transfer of energy is defined separately later.

Active elements
These are elements in which energy can be ‘generated’ (a pseudonym for the transformation of energy from another form). The active elements are considered to be a voltage source $E$, or a
current source $I$, which are symbolically represented as shown in fig. 2.1(d) and 2.1(e).

(a) resistor  (b) inductor  (c) capacitor  (d) voltage source  (e) current source

Fig. 2.1

The most readily understood form of active element is an energy source of constant voltage. For the purpose of analysis it is sometimes convenient to represent this constant voltage source by an equivalent current source and the equivalence between such sources is explained later.

Branch
In an electrical network an element or several elements, passive or active, connected in series between two terminals, is defined as a branch.

Node
A terminal common to two or more branches is defined as a node, and is often represented in diagrams by a dot. In this book a node is represented by a square which may contain an identifying symbol. The voltage at a node measured with respect to a given reference is defined as the nodal voltage.

Mesh
Any arbitrary closed path consisting of branches is defined as a mesh. A hypothetical current which is assumed to flow in a mesh is defined as a mesh current and is symbolically represented as $i$ in fig. 2.2.

Sign convention
The sign convention adopted in this book is shown in fig. 2.3.
1. $E$ indicates a voltage source with positive voltage difference at point 2 measured with respect to point 1.

2. $v$ indicates a positive voltage difference across a branch at point 2 measured with respect to point 1, when a positive current, $i$, flows from point 2 into point 1.

3. $I$ indicates a positive current source with current flowing in the direction of the arrow into node 1.

4. $i$ indicates a positive hypothetical mesh current flowing in the direction of the arrow around a closed loop.
Kirchhoff's laws

Kirchhoff's laws are illustrated in fig.2.4.

It is not easy to remember which law is the first or second. Kirchhoff himself numbered them differently in different parts of his writings. It is easier to associate the summation of currents entering any node with a nodal or current law and the summation of voltage drops round a closed mesh with a mesh or voltage law.

First or nodal or current law:
The algebraic sum of the currents entering any node is zero:

\[ \sum_{k} I_k = 0 \]

Second or mesh or voltage law:
The algebraic sum of voltage drops round a closed mesh is equal and opposite to the algebraic sum of the e.m.f.'s:

\[ \sum_{k} V_k = -\sum_{k} E_k \]

Basic relationships

In each passive element, the voltage difference \( v \) across the element and the current \( i \) flowing through it are related by the fundamental laws,

- Resistor \( v = Ri \)
- Inductor \( v = L \frac{di}{dt} \)
- Capacitor \( v = \frac{1}{C} \int_{0}^{t} i \, dt + V_{(0)} \)

where \( v \) and \( i \) are the instantaneous values of varying voltage and current and \( V_{(0)} \) is the initial capacitor voltage, usually assumed zero for the purpose of simplifying the analysis.
Impedance

In steady-state alternating current theory, where the currents and voltages are assumed to vary sinusoidally with time, impedance is defined as the ratio of complex voltage/complex current. Impedance is a complex quantity comprising resistance and reactance and can be represented by the expression

\[ Z = R + jX \]

where \( X \) is the reactance.

Inductive reactance \( X_L = +j\omega L \)

Capacitive reactance \( X_C = -j\left(\frac{1}{\omega C}\right) \)

The symbol \( \omega \) is used for the angular frequency in radians per second.

The voltage \( v \) across a branch, the current \( i \) through the branch and the impedance \( Z \) of the branch are related by Ohm's law as

\[ Z = \frac{v}{i} \]

where \( v \) and \( i \) are the complex values.

Admittance

The reciprocal of \( Z \) is called admittance and is represented by the symbol \( Y \),

\[ Y = \frac{1}{Z} = \frac{i}{v} \]

Complex conjugate

A complex number which differs from another complex number only in the sign of the imaginary part is called a complex conjugate and is denoted by means of an asterisk thus:

if \( Z = R + jX \) then the complex conjugate is \( Z^* = R - jX \)

The complex conjugate of \( Z^* \) is \( Z \) again.

Complex "power"

Where the voltage \( V \) and the current \( I \) are both real quantities
only, then the product $VI$ gives the real power $P$ in watts. If, however, both $V$ and $I$ are complex then the product $VI^*$ gives the 'power' consisting of two components, the real power $P$ in watts and the reactive power $Q$ in $VAR$. The complex power is represented by $S$ so that

$$S = P \pm jQ$$

the sign of $jQ$ depending upon the sign of the imaginary part of $V$ and $I$.

By convention $jQ$ is assumed positive when the complex current $I$ lags the complex voltage $V$. Using the subscripts $p$ and $q$ to denote the in-phase or real part and the quadrature or imaginary part respectively of the quantities concerned and taking the case when

$$V = (V_p + j0)$$

and $I = (I_p - jI_q)$

the expression $S = P + jQ$

is obtained by multiplying the voltage $V$ by the conjugate current $I^*$.

Thus $S = VI^* = (V_p + j0)(I_p + jI_q) = P + jQ$

Equating real and imaginary components results in

$$P = V_p I_p \quad \text{and} \quad Q = V_p I_q$$

It will be noted that $(I_p + jI_q)$ is the conjugate of $(I_p - jI_q)$ which was the assumed current.

If $jQ$ had been defined positive for leading current then it is easily verified that $S = V^*I$.

**Mutual inductance**

When current-carrying coils are located near one another so that their magnetic fields are coupled, a changing current in one will induce an e.m.f. in the other.

Consider the network shown in fig. 2.5 where $L_1$ and $L_2$ are the coefficients of self-inductance relating the voltage differences in each coil to the current flowing therein and $M_{12}$ and $M_{21}$ are coefficients of mutual inductance relating the e.m.f. induced in one coil to the rate of change of current in the other coil.
Using Kirchhoff's second law the following equations can be established:

\[ L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} = E_1 \]

\[ M_{21} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = -v_2 = -i_2 R \]

In general equivalent circuit analysis this relationship is reciprocal, i.e. \( M_{12} = M_{21} \), and the polarity of the effect, i.e. the sign of \( M \), depends on the physical arrangement of the coils [6].

The winding sense of coils may be indicated by putting the number 1 at a positive end and the number 2 at the negative end so that when both currents enter or leave end 1, \( M \) is positive. Symbolically this can be indicated as shown in fig. 2.6.

An alternative and in some circumstances more easily applicable expression of this convention is as follows:

First define arbitrary positive directions of current in the
separate coils and hence determine the sense of the self-induced e.m.f. Then, if a positive current in one coil produces a mutual e.m.f. in the same sense as the self-induced e.m.f. in the second coil, the mutual coefficient is positive and vice versa.

**Mesh current analysis**

Consider the mesh-current method of solving the electrical network shown in fig. 2.7 in which voltage sources and impedances are known quantities. The object is to solve for the unknown branch currents indicated in the figure by \( i_1, i_2, \ldots, i_6 \).

![Mesh current network diagram](image)

**Fig. 2.7**

In order to establish the equations which describe the performance of the network it is convenient to define a *hypothetical* mesh current circulating in each loop.

**Number of independent mesh currents**

The choice of a closed loop and the direction of the assumed current is quite arbitrary, provided that at least one independent mesh current flows through each element in the network.

The number of independent mesh-currents is related to the number of branches and nodes. The relationship can be derived from topological considerations and is given in its simplest form as [7]

\[
M = B - N + 1
\]

where

- \( M \) = number of independent mesh-currents
- \( B \) = number of branches
- \( N \) = number of nodes.

This relationship holds good for such networks as considered up
Network Analysis

to now and which are characterized by having all nodes interconnected through branches. For more complicated networks, for example with mixed sources or transformers, the relationship is changed as shown in later chapters.

For the network in fig. 2.7, B = 5, N = 3, hence M = 5 - 3 + 1 = 3. It will be noted that the two nodes at the bottom of the figure are in fact one common node only, since there is no active or passive element between them.

Mesh impedance matrix

Using the defined mesh currents, indicated in fig. 2.7 by suffixes p, q, r, the next step is to express the currents in all branches \( (i_1, i_2, ..., i_5) \) in terms of \( i_p, i_q, \) and \( i_r \):

\[
\begin{align*}
  i_1 &= i_p \\
  i_2 &= i_q \\
  i_3 &= i_r \\
  i_4 &= i_p - i_q \\
  i_5 &= i_q - i_r
\end{align*}
\]

The voltage difference across each branch can now be expressed in terms of the defined mesh currents as follows

\[
\begin{align*}
  v_1 &= Z_1 i_p, \quad v_2 = Z_2 i_q, \quad v_3 = Z_3 i_r, \\
  v_4 &= Z_4 (i_p - i_q), \quad v_5 = Z_5 (i_q - i_r).
\end{align*}
\]

Using these voltages and Kirchhoff's second law, the voltage difference equations can be written for each mesh in turn as

\[
\begin{align*}
  Z_1 i_p + Z_4 (i_p - i_q) &= E_p \\
  Z_2 i_q + Z_5 (i_q - i_r) - Z_4 (i_p - i_q) &= 0 \\
  Z_3 i_r - Z_5 (i_q - i_r) &= -E_r
\end{align*}
\]

or

\[
\begin{align*}
  (Z_1 + Z_4) i_p - Z_4 i_q + 0 i_r &= E_p \\
  -Z_4 i_p + (Z_2 + Z_5 + Z_4) i_q - Z_5 i_r &= 0 \\
  0 i_p - Z_5 i_q + (Z_3 + Z_5) i_r &= -E_r
\end{align*}
\]
These equations can be written in matrix form as

\[
\begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & 0 \\
-Z_4 & (Z_2 + Z_5 + Z_4) & -Z_5 \\
0 & -Z_5 & (Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix}
=
\begin{bmatrix}
E_p \\
0 \\
-E_r
\end{bmatrix}
\]

or in condensed form as

\[\mathbf{Z} \mathbf{i} = \mathbf{E}\]

The matrix \(\mathbf{Z}\) is called the *mesh-impedance* matrix. The terms on the principal diagonal (from the top left) are called *self-impedances* and are given by the sum of all impedances around the appropriate mesh. The remaining terms are called *mutual-impedances* and are numerically equal to the impedance which is common to two adjacent mesh currents. Although these mutual effects could arise because of electromagnetic coupling (arising from mutual-inductance) between the meshes, it can be seen that they also relate voltage differences in one mesh to currents in another. The previous use of the term mutual therefore remains consistent. Since the choice of the mesh currents is arbitrary, it follows that for any given problem the elements of \(\mathbf{Z}\) may differ numerically depending on the choice of \(\mathbf{i}\), but the final solution for the branch currents must be the same.

*Mesh impedance matrix by inspection*

In practice the evaluation of the impedance matrix \(\mathbf{Z}\) as described above is laborious and can be avoided by writing down the matrix by inspection [8]. Where there are no electromagnetic mutual coupling effects, the diagonal and non-diagonal elements of the matrix can be written down as follows:

**Main diagonal elements.** The voltage difference due to any mesh current is a function of the sum of the impedances through which that current flows. Therefore any element on the principal diagonal consists of the algebraic sum of the individual branch impedances traversed by the current corresponding to that element. Thus for the network shown in fig. 2.7 the diagonal elements are
Network Analysis

\[
\begin{pmatrix}
(Z_1 + Z_4) & . & . \\
. & (Z_2 + Z_4 + Z_5) & . \\
. & . & (Z_3 + Z_5)
\end{pmatrix}
\]

**Non-diagonal elements.** The voltage difference induced in an element of one mesh by a current in another mesh to which the element is common, is a function of the value of impedance of the common element. Therefore each element in any row and column consists of the algebraic sum of the branch elements which are traversed by any two mesh currents, the one corresponding to the row concerned and the other corresponding to the column concerned. If the currents are in opposite directions through the branch a negative sign must be attached. Thus for the network shown in fig. 2.7, the non-diagonal elements are

\[
\begin{pmatrix}
. & -Z_4 & 0 \\
-Z_4 & . & -Z_5 \\
0 & -Z_5 & .
\end{pmatrix}
\]

It follows from the above that the non-diagonal elements must be symmetrical.

If the impedance matrix is given, the above rules can be applied in reverse to obtain an equivalent network but there may be many such networks which satisfy the given data, i.e. the reverse transformation is not unique.

If electromagnetic mutual coupling effects are present the following rules apply:

**Main diagonal elements.** Any element on the main diagonal consists of the algebraic sum of the individual branch impedances traversed by the current corresponding to that element and all the mutual coefficients between those branches. If the mesh current flows in the opposite winding sense through a branch impedance (i.e. 1 to 2 in one and 2 to 1 in the other), a negative sign must be attached to the mutual coefficients. Thus for the network shown in fig. 2.8, the diagonal elements are
Non-diagonal elements. Each element in any row and column, other than on the main diagonal, consists of the algebraic sum of the branch elements which are traversed by both mesh currents, one corresponding to the row concerned, and the other corresponding to the column concerned, together with mutual coefficients due to the same two mesh currents. If the two currents are opposed in the direction through the branch, negative signs must also be attached to the branch impedances. Negative signs must also be attached to the mutual coefficients if the two mesh currents flow in the opposite winding sense. Thus for the network shown in fig. 2.8 the non-diagonal elements are

\[
\begin{bmatrix}
-Z_4 + M_{12} - M_{14} & 0 \\
-Z_4 + M_{21} - M_{41} & -Z_5 - M_{23} \\
0 & -Z_5 - M_{32}
\end{bmatrix}
\]

Care has to be exercised when formulating equations with mutual coefficients particularly with respect to signs. In chapter 4 a systematic technique is described which reduces the entire process to simple routine steps.
Nodal voltage analysis

In mesh current analysis, the electrical networks can have direct physical counterparts. For example, a voltage source can be a battery or a generator and a branch can be a resistance or the series elements of a transmission line. The advantage of such direct analogy in aiding understanding is obvious but for large networks with complicated interconnections difficulties arise. For example, it is not easy to draw on a network diagram the mesh currents for networks with cross-over branches and the processes for establishing the mesh impedance matrix in such cases are also complicated.

Often the number of variables which are required to describe a given physical problem can be reduced and the formulation process simplified by replacing each voltage source by an equivalent current source.

**Equivalence of sources**

If an energy source is given as a voltage source it can easily be converted into an equivalent nodal current source. The complementary conversion of current sources to equivalent voltage sources is rarely used.

![Diagram](image)

(a) general branch

\[ I_a = -YE \quad I_b = YE \]

(b) equivalent branch

**Fig. 2.9**

Fig. 2.9 (a) shows a general branch with an admittance \( Y \) in series with a voltage source \( E \) connected between nodes a and b.
The voltage difference between the two nodes is of magnitude $v$. The current $i$ flowing into node $a$ and hence through the branch is

$$i = Y(v + E)$$

This can be written as

$$i - YE = Yv$$

or as

$$i + I = Yv$$

Hence the voltage source $E$ can be removed and the same voltage difference $v$ between nodes $a$ and $b$ maintained if the current through branch $Y$ is increased by $I = -YE$. This can be effected by current source $I_a$ at node $a$ of magnitude $I_a = -YE$ and current source $I_b$ at node $b$ of magnitude $I_b = YE$ as shown in fig. 2.9(b).

If one of the two nodes is a reference node then the corresponding Kirchhoff current equation for that node is not used. Accordingly the equivalent current source at that node need not appear on the diagrams.

**Nodal admittance matrix**

Consider the network shown in fig. 2.10 where the admittances shown are the reciprocals of the branch impedances. Denoting

![Diagram](image)

**Fig. 2.10**

the nodal voltages indicated in fig. 2.10 by suffixes $a$, $b$, $c$, currents in all branches can be expressed in terms of these voltages as
Network Analysis

\[ i_1 = Y_1 (v_a - v_b) \]
\[ i_2 = Y_2 (v_b - v_c) \]
\[ i_3 = Y_3 v_a \]
\[ i_4 = Y_4 v_b \]
\[ i_5 = Y_5 v_c \]

Using these currents and Kirchhoff’s first law, the current equations can be written for each node in turn.

\[ Y_1 (v_a - v_b) + Y_3 v_a = I_a \]
\[ Y_2 (v_b - v_c) - Y_1 (v_a - v_b) + Y_4 v_b = I_b \]
\[ Y_5 v_c - Y_2 (v_b - v_c) = I_c \]

These equations in matrix form are

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 & 0 \\
-Y_1 & (Y_1 + Y_2 + Y_4) & -Y_2 \\
0 & -Y_2 & (Y_2 + Y_5)
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
= \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
\]

or in condensed form

\[ Yv = I \]

The solution of these equations, i.e. obtaining the unknowns \( v \) for given \( I \), leads to complete solution of the problem.

The matrix \( Y \) is called the \textit{nodal admittance matrix}. The terms on the diagonal are called the \textit{self-admittances} of the corresponding nodes and are given by the sum of all admittances connected to the appropriate node. The non-diagonal terms are called \textit{mutual admittances} between adjacent nodes and are given by the admittances between the corresponding nodes with a negative sign. The sign follows since all nodal voltages are defined as differences above reference.

The inverse of the admittance matrix \( Y \) is usually denoted by \( Z \):

\[ Z = Y^{-1} \]

The matrix \( Z \) is called the \textit{nodal impedance} matrix. A clear distinction should be observed between the \textit{nodal} impedance and
the mesh impedance matrix. Although the same symbol is employed, the context invariably clarifies the meaning.

The decision whether to use mesh current analysis or nodal voltage analysis requires careful consideration. Factors such as the convenience of formulating the equations, the number of variables involved and the purpose of the analysis are some of the points which must be taken into account. In addition practical experience of the type of problem and of the physical nature of the energy sources, i.e. the approximation to a constant current or a constant voltage source, may also influence the decision. Further discussion of the merits of mesh and nodal analysis is given in later chapters.

_Nodal admittance matrix by inspection_

In practice, the nodal admittance matrix can be written down by inspection of the network more easily than the mesh impedance matrix. Where there are no mutual coupling effects, the diagonal and non-diagonal elements of the matrix can be written down as follows:

**Main diagonal elements.** Any element on the main diagonal consists of the algebraic sum of the individual branch admittances connected to the node corresponding to that element. Thus for the network shown in fig. 2.10, the diagonal elements are

\[
\begin{bmatrix}
(Y_1 + Y_3) & \cdot & \cdot \\
\cdot & (Y_1 + Y_2 + Y_4) & \cdot \\
\cdot & \cdot & (Y_2 + Y_5)
\end{bmatrix}
\]

**Non-diagonal elements.** Each element in any row and column, other than on the main diagonal, consists of the algebraic sum of the branch admittances which are connected between two nodes, one corresponding to the row concerned and the other corresponding to the column concerned. A negative sign is always attached to the element. Thus for the network shown in fig. 2.10, the non-diagonal elements are
The non-diagonal elements are symmetrical and the rules can be applied in reverse to obtain a corresponding equivalent network from a given admittance matrix, but again there is no unique correspondence.

**Mutual coupling effects**

If these effects are present then nodal analysis becomes so complex that its use is rarely economic. It is considered that no useful purpose would be served by considering the subject further in this book.

**Asymmetrical mutual admittances**

So far it has been assumed that all nodal admittance matrices are symmetrical. However, there are many problems which can be expressed by equivalent networks with asymmetrical couplings. Such problems often arise in connection with mechanical engineering or with electronic circuits. The solution of some typical problems of this type is discussed in detail in chapter 6.

In electrical power system analysis it is sometimes necessary to solve nodal equations with asymmetrical coefficients. Such equations arise, for example, from a network in which a phase shifting transformer is represented. The effect of a positive quadrature or imaginary tap of \( q \) per unit on the sending-end side of a phase-shifting transformer can be defined as shown by the ideal equivalent circuit and vector diagram of fig. 2.11 in which an ideal phase-shifting unit is in series with a pure admittance \([9]\).

![Fig. 2.11](image)
Using this definition together with the requirement that complex powers on both sides of the ideal unit should be equal, leads to the following equations:

\[ V_s l_s^* = V' l_r^* \]

By definition
\[ V_s = (1 + jq)V' \]

hence
\[ l_r^* = (1 + jq)l_s^* \]

By taking the conjugate of both sides
\[ l_r = (1 - jq)l_s \]

but
\[ l_r = Y(V' - V_r) \]

hence
\[ l_r = \frac{YV_s}{1 + jq} = YV_r \]  \hspace{1cm} (2.1)

and
\[ l_s = \frac{YV_s}{1 + q^2} - \frac{YV_r}{1 - jq} \]  \hspace{1cm} (2.2)

Expressing equations (2.2) and (2.1) in general terms
\[ Y_{ss} V_s - Y_{sr} V_r = I_s \]
\[ Y_{rs} V_s - Y_{rr} V_r = I_r \]

or in matrix form
\[
\begin{bmatrix}
Y_{ss} & -Y_{sr} \\
Y_{rs} & -Y_{rr}
\end{bmatrix}
\begin{bmatrix}
V_s \\
V_r
\end{bmatrix}
= \begin{bmatrix}
I_s \\
I_r
\end{bmatrix}
\]

where
\[ Y_{ss} = \frac{Y}{1 + q^2}, \quad Y_{sr} = \frac{Y}{1 - jq}, \]
\[ Y_{rs} = \frac{Y}{1 + jq}, \quad Y_{rr} = Y \]

The problem is characterized by the fact that the admittance matrix is not symmetrical since \( Y_{sr} \neq Y_{rs} \). The transformer with quadrature tap can be represented by an equivalent network with unilateral mutual coupling but it is more convenient to replace this network by another equivalent network as shown in fig. 2.12 in which the voltage source \( V_s \) in series with branch admittance
$Y_3$ is such that its e.m.f. is always equal to the nodal voltage $V_s$.

The values of the individual branch admittances can be established from fig. 2.12 and equations (2.1) and (2.2).

\[
\begin{align*}
(Y_1 + Y_2)V_s - Y_1 V_r &= I_s \\
(Y_1 + Y_3)V_s - (Y_1 + Y_3 + Y_4)V_r &= I_r
\end{align*}
\]

Therefore

\[
\begin{align*}
Y_{ss} &= Y_1 + Y_2 \\
Y_{sr} &= Y_1 \\
Y_{rs} &= Y_1 + Y_3 \\
Y_{rr} &= Y_1 + Y_3 + Y_4
\end{align*}
\]

Hence

\[
\begin{align*}
Y_1 &= Y_{sr} = \frac{Y}{1-jq} \\
Y_2 &= Y_{ss} - Y_1 = \frac{Y}{1+q^2} - \frac{Y}{1-jq} = -jqY \\
Y_3 &= Y_{rs} - Y_1 = \frac{Y}{1+jq} - \frac{Y}{1-jq} = -2jqY \\
Y_4 &= Y_{rr} - Y_1 - Y_3 = Y - \frac{Y}{1-jq} - \frac{Y}{1+jq} + \frac{Y}{1-jq} = Y - \frac{Y}{1+jq} = jqY
\end{align*}
\]

**Mixed networks**

Networks in which the energy sources are represented by nodal currents and voltage sources as shown in fig. 2.12, are defined as mixed networks [10].

In general, any sets of linear equations in which the coefficient
matrix is not symmetrical can be represented by an equivalent mixed network in which the voltage source is a function of the unknown nodal voltage. Special consideration is given to the solution of mixed networks in chapter 6.

Problems
1. Write the mesh current equations in matrix form for the networks shown in
   (a) fig. 2.13, (b) fig. 2.14, (c) fig. 2.15.
2. Draw equivalent nodal networks for
   (a) fig. 2.16, (b) fig. 2.17

Fig. 2.15

Fig. 2.16
3. Write the nodal voltage equations in matrix form for the networks shown in
   (a) fig. 2.18, (b) fig. 2.19, (c) fig. 2.20.
CHAPTER THREE
Solution of Network Equations

The linear equations which describe the performance of an electrical network can be written in the mesh current or the nodal voltage form as shown in the previous chapter. Having set up these equations, probably using matrix methods if the problem is at all complex, consideration should be given to the best methods of solution. In this sense 'best' usually means the method requiring the fewest operations to achieve the required accuracy but, when digital computers are involved, it may also be desirable to use those methods requiring least storage space.

There are many methods for the numerical solution of linear simultaneous equations; some are more applicable for calculations using, say, a slide-rule or a desk calculator while others are more suitable for a digital computer. In this chapter some of the methods which are also particularly applicable to the solution of electrical network equations are examined. The theory of these methods is presented in some detail so that their relative merits can be understood, but the reader is referred to standard texts on numerical methods [1, 2, 3, 4, 5, 11] for such items as automatic checking procedures or reduction of errors due to round-off.

Since it has been shown in chapter 1 that equations involving complex quantities can be reduced to all-real equations, no special attention is given to such problems. However, in the more general descriptions, it may be seen that it is equally convenient to regard each operation as one in complex arithmetic and many digital computer subroutines have been developed in this way.

Sometimes, for reasons explained in this chapter, some equations may be inherently more difficult to solve than others. In some such cases special measures can be taken. In general, three types of solution may be recognized: direct, iterative and hybrid.

Direct methods can be defined as those which give a direct solution, within the accuracy of working, after a fixed number of operations. In practice these methods usually occupy large computer storage space and take a long time.

Iterative methods require less space and give results in a series of approximations of increasing accuracy. Each approxima-
tion starts from the previous solution and involves less time than
the direct method although the overall solution may take longer,
depending on the accuracy required and the nature of the equa-
tions. In practice not all equations can be solved by these meth-
ods.

Hybrid methods are combinations of the direct and iterative
methods. Networks can be subdivided into a number of smaller
sub-networks; each part can be solved in principle by a direct
method and the partial solutions combined in a series of approxi-
mations to obtain the final solution. In practice these methods
are more suitable for large networks or for non-linear problems.

Successive elimination

The most elementary direct method of solving a set of simultan-
eous linear equations is by systematic elimination of the vari-
bles. The basic step of the method consists of elimination of
one variable at a time. Consider a set of simultaneous equations
in \( n \) variables

\[
\begin{align*}
A_{11}X_1 + A_{12}X_2 + & \quad \cdots \quad + A_{1n}X_n = B_1 \\
A_{21}X_1 + A_{22}X_2 + & \quad \cdots \quad + A_{2n}X_n = B_2 \\
\vdots & \quad \vdots \\
A_{n1}X_1 + A_{n2}X_2 + & \quad \cdots \quad + A_{nn}X_n = B_n
\end{align*}
\]

The first equation can be used to eliminate \( X_1 \) from the remain-
ing equations resulting in \((n - 1)\) equations in \((n - 1)\) variables.
A convenient way is to multiply the first equation by \( A_{r1}/A_{11} \),
and subtract it from the \( r \)th equation. The element \( A_{11} \) is called
the pivot and the equation containing the pivot is called the
pivotal equation and remains unchanged.

The basic step in the elimination method is repeated to elimi-
nate the remaining variables until one equation with one unknown
is left. The various pivotal equations can then be assembled
together in the triangular form

\[
\begin{align*}
C_{11}X_1 + C_{12}X_2 + \cdots + C_{1n}X_n &= D_1 \\
C_{22}X_2 + \cdots + C_{2n}X_n &= D_2 \\
\vdots & \quad \vdots \\
C_{nn}X_n &= D_n
\end{align*}
\]
The last equation containing one unknown can be solved directly and
the remaining variables follow in turn by substituting into the
triangular form in reverse order, known as back-substitution.

The process of elimination and back-substitution is known as
*Gaussian elimination* or *pivotal condensation*. The operations
involved in this process have been reduced to an absolute mini-
mum and automatic check procedures have been incorporated in a
method specially suitable for desk machines known as Crout's
process [4]. The method is not suitable for large problems.

**Elimination of variables and network reduction**

Sometimes, in network analysis, the solution for all the variables
together is of no immediate interest. When this is the case, the
amount of calculation can be reduced by eliminating the variables
for which solutions are not required.

Consider the following set of nodal voltage equations

\[
\begin{align*}
Y_{11} v_1 & + Y_{13} v_3 + Y_{14} v_4 = I_1 \\
Y_{22} v_2 & + Y_{25} v_5 = I_2 \\
Y_{31} v_1 & + Y_{33} v_3 + Y_{35} v_5 = I_3 \\
Y_{41} v_1 & + Y_{44} v_4 = I_4 \\
Y_{52} v_2 + Y_{53} v_3 & + Y_{55} v_5 = I_5
\end{align*}
\]

where \(v_1, \ldots, v_5\) are the unknown voltages. Suppose that the solu-
tion for the last two voltages, \(v_4\) and \(v_5\), is not required. Writing
the same equations in a matrix form and partitioning the variables
for which a solution is required from the remainder,

\[
\begin{bmatrix}
Y_{11} & Y_{13} & Y_{14} & 0 \\
0 & Y_{22} & 0 & Y_{25} \\
Y_{31} & 0 & Y_{33} & 0 \\
Y_{41} & 0 & 0 & Y_{44} \\
0 & Y_{52} & Y_{53} & 0 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
\end{bmatrix}
= \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
Y_{11} & Y_{13} & Y_{14} \\
0 & Y_{22} & 0 \\
Y_{31} & 0 & Y_{33} \\
Y_{41} & 0 & 0 \\
0 & Y_{52} & Y_{53} \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
\end{bmatrix}
= \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
Y_{11} & Y_{13} & Y_{14} \\
0 & Y_{22} & 0 \\
Y_{31} & 0 & Y_{33} \\
Y_{41} & 0 & 0 \\
0 & Y_{52} & Y_{53} \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
\end{bmatrix}
= \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
\end{bmatrix}
\]
or in a matrix compound form as

\[
\begin{bmatrix}
Y_{11} & Y_{12} \\
\vdots & \vdots \\
Y_{21} & Y_{22}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= 
\begin{bmatrix}
l_1 \\
l_2
\end{bmatrix}
\]

where

\[
Y_{11} = 
\begin{bmatrix}
Y_{11} & 0 & Y_{13} \\
0 & Y_{22} & 0 \\
Y_{13} & 0 & Y_{13}
\end{bmatrix}
\quad Y_{12} = 
\begin{bmatrix}
Y_{14} & 0 \\
0 & Y_{25} \\
0 & Y_{25}
\end{bmatrix}
\quad v_1 = 
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
\quad l_1 = 
\begin{bmatrix}
l_1 \\
l_2 \\
l_3
\end{bmatrix}
\]

\[
Y_{21} = 
\begin{bmatrix}
Y_{41} & 0 & 0 \\
0 & Y_{52} & Y_{53}
\end{bmatrix}
\quad Y_{22} = 
\begin{bmatrix}
Y_{44} & 0 \\
0 & Y_{55}
\end{bmatrix}
\quad v_2 = 
\begin{bmatrix}
v_4 \\
v_5
\end{bmatrix}
\quad l_2 = 
\begin{bmatrix}
l_4 \\
l_5
\end{bmatrix}
\]

The matrix \( Y \) has been partitioned to give consistent multiplications, thus \( Y_{11} \) and \( Y_{22} \) are square matrices.

Expanding the compound matrix equation

\[
Y_{11} v_1 + Y_{12} v_2 = l_1
\]
\[
Y_{21} v_1 + Y_{22} v_2 = l_2
\]

Solving for \( v_2 \), from the second equation

\[
Y_{22} v_2 = l_2 - Y_{21} v_1
\]
\[
\therefore \quad v_2 = Y_{22}^{-1}(l_2 - Y_{21} v_1)
\]

Substituting for \( v_2 \) into the first equation

\[
Y_{11} v_1 + Y_{12} Y_{22}^{-1}(l_2 - Y_{21} v_1) = l_1
\]
\[
\therefore \quad (Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}) v_1 = l_1 - Y_{12} Y_{22}^{-1} l_2
\]

The last equation can be written as

\[
Y_{11}' v_1 = l_1'
\]

where

\[
Y_{11}' = Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}
\]

\[
l_1' = l_1 - Y_{12} Y_{22}^{-1} l_2
\]

The solution for \( v_1 \) can now be obtained by inverting \( Y_{11}' \)

\[
v_1 = Z_{11}' l_1'
\]
where \( Z'_{11} = (Y'_{11})^{-1} \)

In many cases it may be convenient to partition quantities corresponding to nodes with current sources from those without current sources; thus if \( l_2 = 0 \), then \( l'_1 = l_1 \) and the solution is

\[ v_1 = Z'_{11} l_1 \]

\( Y'_{11} = Y_{11} - Y_{12} Y_{22}^{-1} Y_{21} \), is called the reduced admittance matrix and contains as many rows and columns as the number of variables not eliminated.

The computation is simplified if the reduction is carried out eliminating one variable at a time rather than by eliminating a set of variables and it is then equivalent to the well-known process of network reduction by star-delta transformations which eliminate one node at a time.

An equivalent reduced network can be constructed from the reduced admittance matrix \( Y'_{11} \) and the equation \( Y'_{11} v_1 = l'_1 \), as shown in the following numerical example.

**Numerical example**

As an example consider first the full solution of the network shown in fig. 3.1(a).

![Original and reduced network](image)

Fig. 3.1

The equation \( Yv = l \) can be constructed by inspection as

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
 v_a \\
v_b \\
v_c
\end{bmatrix}
=
\begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix}
\]
The inverse of $\mathbf{Y}$ is

$$
\mathbf{Z} = \frac{1}{8} \begin{bmatrix}
5 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 5
\end{bmatrix}
$$

The full solution can be obtained from

$$
\mathbf{v} = \mathbf{Zl}
$$

$$
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix} = \frac{1}{8} \begin{bmatrix}
5 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 5
\end{bmatrix} \begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix} = \frac{1}{8} \begin{bmatrix}
17 \\
10 \\
13
\end{bmatrix}
$$

Now suppose that from the outset the voltage $v_b$ is not required, then the three rows and columns of the original admittance matrix can be reduced to two.

Rewriting the equations so that the equation of the unwanted variable comes last and partitioning,

$$
\begin{bmatrix}
2 & 0 & \vdots & -1 \\
0 & 2 & \vdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \vdots & 3
\end{bmatrix} \begin{bmatrix}
v_a \\
v_c \\
v_b
\end{bmatrix} = \begin{bmatrix}
3 \\
2 \\
0
\end{bmatrix}
$$

Let the four parts of the admittance matrix be labelled

$$
\mathbf{Y} = \begin{bmatrix}
\mathbf{Y}_{11} & \mathbf{Y}_{12} \\
\vdots & \vdots \\
\mathbf{Y}_{21} & \mathbf{Y}_{22}
\end{bmatrix}
$$

Then evaluating $\mathbf{Y}'_{11}$ from

$$
\mathbf{Y}'_{11} = \mathbf{Y}_{11} - \mathbf{Y}_{12} \mathbf{Y}_{22}^{-1} \mathbf{Y}_{21}
$$

$$
\mathbf{Y}_{22}^{-1} \mathbf{Y}_{21} = \frac{1}{3} \begin{bmatrix}
-1 & -1 \\
-1 & -1
\end{bmatrix}
$$

$$
\mathbf{Y}_{12} \mathbf{Y}_{22}^{-1} \mathbf{Y}_{21} = \frac{1}{3} \begin{bmatrix}
-1 \\
-1
\end{bmatrix} \begin{bmatrix}
-1 & -1 \\
-1 & -1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
$$
\[ Y'_{11} = Y_{11} - Y_{12} Y^{-1}_{22} Y_{21} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \]

Hence, \( Y'_{11} v_1 = I'_{1} \) is

\[
\frac{1}{3} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} v_a \\ v_c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

An equivalent reduced network can be constructed from \( Y'_{11} \) and \( I_1 \) \((I_2 = 0)\) and the most simple network is shown in fig.3.1(b). The inverse of \( Y'_{11} \) is

\[
Z'_{11} = \frac{1}{5} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}
\]

The required solution can now be obtained from

\[
v_1 = Z'_{11} I_{11}
\]

\[
\begin{bmatrix} v_a \\ v_c \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 17 \\ 13 \end{bmatrix}
\]

For simplicity of explanation, in this example the equations of the problem have been rearranged before elimination. In practice such rearrangements can be taken into account in the routine of the method.

**Inverse of a matrix**

The inverse of a matrix may be required for many reasons. It may be required to solve the equation

\[ AX = B \]

for the unknown \( X \) a number of times, with the same coefficient matrix \( A \) but with different matrices \( B \). It is obviously most convenient to obtain the inverse matrix \( A^{-1} \) and then solve for each \( X \) from

\[ X = A^{-1} B \]

Many technological investigations require a number of solutions with the same or different matrix \( B \) and small changes in the coefficient matrix \( A \). Such problems can also be conveniently solved by obtaining the inverse matrix \( A^{-1} \) and modifying it as
shown later in chapter 7.

One of the simplest and most efficient methods of calculating the inverse of large matrices is an extension of the process of elimination of variables as described in the previous section.

The method can be illustrated by considering the solution of a general set of equations in \( n \) variables, \( AX = B \), where the elements can be real or complex numbers and the coefficients symmetrical or asymmetrical. For example in the set of equations having 5 unknowns,

\[
\begin{align*}
A_{11}X_1 + A_{12}X_2 + A_{13}X_3 + A_{14}X_4 + A_{15}X_5 &= B_1 \\
A_{21}X_1 + A_{22}X_2 + A_{23}X_3 + A_{24}X_4 + A_{25}X_5 &= B_2 \\
A_{31}X_1 + A_{32}X_2 + A_{33}X_3 + A_{34}X_4 + A_{35}X_5 &= B_3 \\
A_{41}X_1 + A_{42}X_2 + A_{43}X_3 + A_{44}X_4 + A_{45}X_5 &= B_4 \\
A_{51}X_1 + A_{52}X_2 + A_{53}X_3 + A_{54}X_4 + A_{55}X_5 &= B_5 \\
\end{align*}
\]

where \( X_1, \ldots, X_5 \) are the unknowns. Writing the equations in matrix form and partitioning arbitrarily into two groups, for example,

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{14} & A_{15} \\
A_{21} & A_{22} & A_{23} & \cdots & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & \cdots & A_{34} & A_{35} \\
A_{41} & A_{42} & A_{43} & \cdots & A_{44} & A_{45} \\
A_{51} & A_{52} & A_{53} & \cdots & A_{54} & A_{55}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\cdots \\
X_5
\end{bmatrix} =
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\cdots \\
B_5
\end{bmatrix}
\]

or in a compound matrix form as

\[
\begin{bmatrix}
A_{11} & \cdots & A_{12} \\
\cdots & \cdots & \cdots \\
A_{21} & \cdots & A_{22}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} =
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
\]

(3.1)

The coefficient matrix is partitioned in such a manner that \( A_{11} \) and \( A_{22} \) are square matrices.

Expanding the compound matrix equation,

\[
\begin{align*}
A_{11}X_1 + A_{12}X_2 &= B_1 \\
A_{21}X_1 + A_{22}X_2 &= B_2 \\
\end{align*}
\]
Eliminating \( X_2 \) from the second equation,

\[
A_{22}X_2 = B_2 - A_{21}X_1
\]

\[
\therefore X_2 = A_{22}^{-1}B_2 - A_{22}^{-1}A_{21}X_1
\]  \hspace{1cm} (3.2)

Substituting for \( X_2 \) in the first equation,

\[
A_{11}X_1 + A_{12}A_{22}^{-1}B_2 - A_{12}A_{22}^{-1}A_{21}X_1 = B_1
\]

\[
\therefore (A_{11} - A_{12}A_{22}^{-1}A_{21})X_1 + A_{12}A_{22}^{-1}B_2 = B_1
\]  \hspace{1cm} (3.3)

Equations (3.3) and (3.2) can be written together in matrix form as

\[
\begin{bmatrix}
(A_{11} - A_{12}A_{22}^{-1}A_{21}) & A_{12}A_{22}^{-1} \\
-A_{22}^{-1}A_{21} & A_{22}^{-1}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
X_2
\end{bmatrix}
\]

or as

\[
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
X_2
\end{bmatrix}
\]

where \( C_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad C_{12} = A_{12}A_{22}^{-1} \)

\( C_{21} = -A_{22}^{-1}A_{21}, \quad C_{22} = A_{22}^{-1} \)

The above equations are of the same form as the original equations with \( X_2 \) and \( B_2 \) interchanged.

Multiplying out again,

\[
C_{11}X_1 + C_{12}B_2 = B_1
\]

\[
C_{21}X_1 + C_{22}B_2 = X_2
\]

Eliminating this time \( X_1 \) from the first equation,

\[
C_{11}X_1 = B_1 - C_{12}B_2
\]

\[
\therefore X_1 = C_{11}^{-1}B_1 - C_{11}^{-1}C_{12}B_2
\]  \hspace{1cm} (3.4)

Substituting for \( X_1 \) in the second equation,

\[
C_{21}C_{11}^{-1}B_1 - C_{21}C_{11}^{-1}C_{12}B_2 + C_{22}B_2 = X_2
\]

\[
\therefore C_{21}C_{11}^{-1}B_1 + (C_{22} - C_{21}C_{11}^{-1}C_{12})B_2 = X_2
\]  \hspace{1cm} (3.5)

Equations (3.4) and (3.5) can be written together in matrix form as
Solution of Network Equations

\[
\begin{bmatrix}
C_{11}^{-1} & -C_{11}^{-1}C_{12} \\
C_{21}C_{11}^{-1} & (C_{22} - C_{21}C_{11}^{-1}C_{12})
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
= 
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
\]
or as

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
= 
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
\]

or in general form as

\[
DB = X
\]

Comparing equation (3.6) with the original equation (3.1) it is evident that \( D \) is the inverse of \( A \):

\[
D = A^{-1}
\]
The same result could have been obtained if the process was carried out in different order by eliminating \( X_1 \), first and then \( X_2 \).

General systematic method

In the above derivation the original set of equations could have been partitioned into a number of groups and the process repeated a number of times. For example, if the original set of equations were partitioned into three groups according to the following scheme

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}
\]

so that the submatrices on the principal diagonal are all square matrices, then eliminating \( X_1, X_2, X_3 \) respectively, the following matrices can be established:
The final solution is now obtained from

\[ X_1 = E_{11} B_1 + E_{12} B_2 + E_{13} B_3 \]
\[ X_2 = E_{21} B_1 + E_{22} B_2 + E_{23} B_3 \]
\[ X_3 = E_{31} B_1 + E_{32} B_2 + E_{33} B_3 \]

The computation is simplified if the elimination process is applied to one variable at a time, rather than to a set of variables, and repeated as many times as there are variables. This leads to a set of simple rules for the inversion of a matrix.

Select a pivotal element \( A_{pp} \) and carry out the following four steps

1. \( A'_{pp} = \frac{1}{A_{pp}} \)
2. \( A'_{rp} = A_{rp} A'_{pp} \)
3. \( A'_{rc} = A_{rc} - A'_{rp} A_{pc} \)

for all elements except \( A_{rp} \) in row \( r \), and repeat steps (2) and (3) for all remaining rows except the pivotal row.
Solution of Network Equations

(4) \[ A'_{pc} = -A'_{pp}A_{pc} \]

for all elements in row \( p \) except the pivotal element, where \( A_{pp} \) = the pivotal element in row and column \( p \) \( A_{rc} \) = the element in row \( r \) and column \( c \) \( A' \) = the derived elements which replace their predecessors.

The process is repeated for all diagonal terms in the matrix, each repetition starting from the result of the previous one. The resulting matrix is the inverse of the original matrix. The whole process may be carried out in any order.

**Practical steps**

The practical steps are best illustrated by considering a numerical example of finding the inverse of the following matrix.

\[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 3 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

(1) Select any one element on the principal diagonal as pivot and replace it by its reciprocal. For example, if the diagonal element in the second row and column is selected as pivot (printed in bold above), then step (1) will result in

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(2) Select any one row not containing the pivotal element. Multiply the element in the same column as the pivot by the new value of the pivotal element. For example, selecting the third row will result in
\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & -\frac{1}{2} & \cdot & \cdot \\
\cdot & \cdot & -\frac{1}{2} & \cdot \\
\cdot & \cdot & \cdot & \cdot 
\end{bmatrix}
\]

since \((-1) \left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right).

(3) To compute the remaining terms in the selected row \((r)\) take each element in turn, e.g. the element in column \(c\), and subtract from it the product of two elements; one from the selected row and the pivotal column evaluated in step (2), and the other in the pivotal row and column \(c\). This replaces the original element of row \(r\) and column \(c\). Numerically for the element in the third row and third column it is \(3 - \left(-\frac{1}{2}\right)(-1) = 2\frac{1}{2}\).

\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{2} & \cdot & \cdot \\
-\frac{1}{2} & -\frac{1}{2} & 2\frac{1}{2} & -1 \\
\cdot & \cdot & \cdot & \cdot 
\end{bmatrix}
\]

Repeat steps (2) and (3) for all rows except the one containing the pivotal element.

(4) Multiply each element in the pivotal row, except the pivot, by the new value of the pivotal element and change the sign

\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\cdot & \cdot & \cdot & \cdot 
\end{bmatrix}
\]

For full inversion, steps (1) to (4) must be repeated for all diagonal terms.
The complete inversion of the above matrix by this method is summarized as follows, choosing the elements on the leading diagonal as pivots in an arbitrary order:

Selecting the second diagonal element as pivot results in

\[
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2} & 1 & 1 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

Selecting the fourth diagonal element as pivot results in

\[
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2} & 1 & 1 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Selecting the first diagonal element as pivot results in

\[
\begin{bmatrix}
2 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
-1 & -1 & 1 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Selecting the third diagonal element as pivot results in the final inverse which is the same irrespective of the order in which the pivotal elements have been selected.

\[
A^{-1} = \begin{bmatrix}
3 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2
\end{bmatrix}
\]

**General comments**

If at any stage of computation all the remaining diagonal terms are found to be zero, the method has broken down, although an inverse may still exist. This difficulty can be overcome by an
interchange of the order in which the equations are first written down. Although the rules have been stated in such a form that the pivot is always selected from the leading diagonal, this is not essential. Any element in the matrix can be selected as pivot provided that only one element is selected from each row and column. However, in practice this would increase the difficulty of programming the computer. In a general-purpose matrix-inversion procedure it is necessary to make provision for such selection.

For practical networks the possibility of all the remaining diagonal elements being zero at any stage cannot occur. Comparing the steps of inversion described in this section with the steps of network reduction described in the previous section, it is evident that the two are identical. It follows, therefore, that at any stage in the process of inversion, the remaining diagonal terms and associated rows and columns represent the matrix of an equivalent reduced network derived from the original network. If there were non-zero diagonal terms in the coefficient matrix of the original network corresponding to connections to the reference node then there must be similar terms at all stages in the reduced networks.

The inversion of the reduced matrix corresponds to the solution of the reduced network. From this, it follows that the elements of the full inverse of the nodal admittance matrix in the rows and columns corresponding to the node numbers of the reduced network are the elements of the inverse admittance matrix of that reduced network.

![Diagram of network](Fig. 3.2)
Solution of Network Equations

For example, if admittance matrix $\mathbf{Y}$, of the network shown in Fig. 3.2 is

$$
\mathbf{Y} = \begin{bmatrix}
Y_{aa} & Y_{ab} & 0 & 0 & 0 \\
Y_{ba} & Y_{bb} & Y_{bc} & Y_{bd} & 0 \\
0 & Y_{cb} & Y_{cc} & Y_{cd} & 0 \\
0 & Y_{db} & Y_{dc} & Y_{dd} & Y_{de} \\
0 & 0 & 0 & Y_{ed} & Y_{ee}
\end{bmatrix}
$$

and $\mathbf{Z}$, the full inverse of $\mathbf{Y}$ is

$$
\mathbf{Z} = \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} & Z_{ae} \\
Z_{ba} & Z_{bb} & Z_{bc} & Z_{bd} & Z_{be} \\
Z_{ca} & Z_{cb} & Z_{cc} & Z_{cd} & Z_{ce} \\
Z_{da} & Z_{db} & Z_{dc} & Z_{dd} & Z_{de} \\
Z_{ea} & Z_{eb} & Z_{ec} & Z_{ed} & Z_{ee}
\end{bmatrix}
$$

The inverse $\mathbf{Z}'$ of an equivalent network reduced to nodes $a$, $c$, $e$, may be obtained by inspection of $\mathbf{Z}$ as

$$
\mathbf{Z}' = \begin{bmatrix}
Z_{aa} & Z_{ac} & Z_{ae} \\
Z_{ca} & Z_{cc} & Z_{ce} \\
Z_{ea} & Z_{ec} & Z_{ee}
\end{bmatrix}
$$

and so on.

By carrying out the process of inversion in the order of largest coefficient first, the accuracy is improved by reducing the occurrence of numerically large multipliers. The accuracy is further improved if the order in which the equations are written is changed so that the largest elements are on the diagonal. However, for practical networks it is seldom necessary to take such precautions.

The inverse of a symmetrical matrix is another symmetrical matrix. The intermediate steps in the process of elimination do not produce symmetrical matrices but the difference is only in sign. This can very easily be taken care of in a digital computer program and the amount of calculation can be reduced by almost
half by making the calculations for one half only.

The amount of calculation is further reduced by taking advantage of zero elements. For example, if in step (2) \( A'_{rd} \), the product \( A_{rd}A'_{dd} \) is also zero and hence all the elements in the row of \( A_{rd} \) remain unchanged. For this reason when carrying out hand calculations it is best to choose pivots in rows containing the maximum number of zero elements.

The method described makes a minimum demand on storage space in a computer. The particular advantages for hand calculation lie in the fact that it involves a minimum number of different equations and a minimum number of numerical calculations for each equation.

**Factorized inverse table**

The method of successive elimination can be combined with the partitioning method to obtain a factorized inverse table [12]. This is a systematic process for solving linear equations by partitioning the matrix and operating on the submatrices. The process involves the calculation of the inverses of the submatrices on the leading diagonal and other matrix multiplications and additions. The process is the same as the pivotal condensation [2] or Gaussian elimination method but operating on matrices and vectors instead of single elements with the results tabulated systematically in a convenient manner.

The method is useful when the full inverse is not required and when the number of variables becomes larger than is convenient to manipulate simultaneously.

Consider the simplest case of partitioning a set of equations into two groups according to the following scheme:

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
\]

Multiplying out,

\[
A_{11}X_1 + A_{12}X_2 = B_1 \\
A_{21}X_1 + A_{22}X_2 = B_2
\]

From the first equation, assuming the inverse of \( A_{11} \) exists,

\[
X_1 = A_{11}^{-1}(B_1 - A_{12}X_2)
\]
Substituting this in the second,
\[ A_{21} A_{11}^{-1} (B_1 - A_{12} X_2) + A_{22} X_2 = B_2 \]
or
\[ X_2 = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} (B_2 - A_{21} A_{11}^{-1} B_1) \]

let
\[ D_{11} = A_{11}^{-1} \]
\[ C_{12} = D_{11} A_{12} \]
\[ D_{22} = (A_{22} - A_{21} C_{12})^{-1} \]

then
\[ X_1 = D_{11} B_1 - C_{12} X_2 \]
\[ X_2 = D_{22} (B_2 - A_{21} D_{11} B_1) \]

**Practical steps**

For the purpose of systematic calculations it is convenient to construct a table by replacing the elements of \( A_{11} \) from the original partitioned matrix by \( D_{11} \), and those of \( A_{12} \) by \( C_{12} \) and those of \( A_{22} \) by \( D_{22} \), leaving the remaining element \( A_{21} \) unchanged according to the following scheme:
\[
\begin{bmatrix}
  D_{11} & C_{12} \\
  A_{21} & D_{22}
\end{bmatrix}
\]

Denoting intermediate values of \( A_{22} \) by \( A_{22}^{(1)} \), etc. the construction of the factorized inverse table can be calculated in the following steps:

\[ D_{11} = A_{11}^{-1} \]
\[ C_{12} = D_{11} A_{12} \]
\[ A_{22}^{(1)} = A_{22} - A_{21} C_{12} \]
\[ D_{22} = (A_{22}^{(1)})^{-1} \]

The solution for \( X_1 \) and \( X_2 \) can be obtained in reverse order from

\[ X_1^{(1)} = D_{11} B_1 \]
\[ B_2^{(1)} = B_2 - A_{21} X_1^{(1)} \]
\[ X_2 = D_{22} B_2^{(1)} \]
\[ X_1 = X_1^{(1)} - C_{12} X_2 \]
As a further example consider the partitioning of a general set of equations into four groups, thus

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix}
\]

Using the successive elimination method the rows and columns can be eliminated one at a time starting from the first row and column. The original matrix can then be replaced by new elements according to the following scheme:

\[
\begin{bmatrix}
D_{11} & C_{12} & C_{13} & C_{14} \\
A_{21} & D_{22} & C_{23} & C_{24} \\
A_{31} & A_{32}^{(1)} & D_{33} & C_{34} \\
A_{41} & A_{42}^{(1)} & A_{43}^{(2)} & D_{44}
\end{bmatrix}
\]

The elements of the factorized inverse table can be calculated in the following steps:

\[
D_{11} = A_{11}^{-1} \\
C_{12} = D_{11} A_{12} \\
C_{13} = D_{11} A_{13} \\
C_{14} = D_{11} A_{14} \\
A_{22}^{(1)} = A_{22} - A_{21} C_{12} \\
A_{23}^{(1)} = A_{23} - A_{21} C_{13} \\
A_{24}^{(1)} = A_{24} - A_{21} C_{14} \\
A_{32}^{(1)} = A_{32} - A_{31} C_{12} \\
A_{33}^{(1)} = A_{33} - A_{31} C_{13} \\
A_{34}^{(1)} = A_{34} - A_{31} C_{14} \\
A_{42}^{(1)} = A_{42} - A_{41} C_{12} \\
A_{43}^{(1)} = A_{43} - A_{41} C_{13}
\]

\[
A_{44}^{(1)} = A_{44} - A_{41} C_{14} \\
D_{22}^{(1)} = (A_{22}^{(1)})^{-1} \\
C_{23}^{(1)} = D_{22} A_{23}^{(1)} \\
C_{24}^{(1)} = D_{22} A_{24}^{(1)} \\
A_{33}^{(2)} = A_{33} - A_{32} C_{23} \\
A_{34}^{(2)} = A_{34} - A_{32} C_{24} \\
A_{43}^{(2)} = A_{43} - A_{42} C_{23} \\
A_{44}^{(2)} = A_{44} - A_{42} C_{24} \\
D_{33}^{(2)} = (A_{33}^{(2)})^{-1} \\
C_{34}^{(2)} = D_{33} A_{34}^{(2)} \\
A_{44}^{(3)} = A_{44} - A_{43} C_{34} \\
D_{44}^{(3)} = (A_{44}^{(3)})^{-1}
\]
The solution can be obtained from the following steps:

\[
\begin{align*}
B_1^{(1)} &= B_1 \\
X_1^{(1)} &= D_{11} B_1^{(1)} \\
B_2^{(1)} &= B_2 - A_{21} X_1^{(1)} \\
X_2^{(1)} &= D_{22} B_2^{(1)} \\
B_3^{(1)} &= B_3 - A_{31} X_1^{(1)} - A_{32} X_2^{(1)} \\
X_3^{(1)} &= D_{33} B_3^{(1)} \\
B_4^{(1)} &= B_4 - A_{41} X_1^{(1)} - A_{42} X_2^{(1)} - A_{43} X_3^{(1)} \\
X_4^{(1)} &= D_{44} B_4^{(1)} \\
X_4 &= X_4^{(1)} \\
X_3 &= X_3^{(1)} - C_{34} X_4 \\
X_2 &= X_2^{(1)} - C_{23} X_3 - C_{24} X_4 \\
X_1 &= X_1^{(1)} - C_{12} X_2 - C_{13} X_3 - C_{14} X_4
\end{align*}
\]

Consider first the solution of a set of linear equations by inverting the coefficient matrix after partitioning and by means of the compound matrix inversion techniques described in pp 54 to 54. Table 3.1 compares the number of operations involved in this process with those required for the solution of the same equation by the factorized inverse table method. For purposes of comparison it is assumed in each case that the coefficient matrix is asymmetrical, contains no zero elements and is divided into equal \( N \times N \) submatrices. Although most networks give rise to symmetrical coefficient matrices, table 3.1 is, nevertheless, a reasonable guide to comparative effectiveness of the two methods.

When a matrix is partitioned, its factorized inverse table has two main advantages over its full inverse:

1. Both methods need the same number of inversions. The full inverse requires a large number of matrix multiplications. The inverse table requires fewer multiplications.

2. For many electrical problems the original matrix contains a large proportion of zero elements. The full inverse has none. The inverse table may contain many zero submatrices.
TABLE 3.1
Comparison of solution by matrix inversion and factorized inverse table

<table>
<thead>
<tr>
<th>Operation</th>
<th>Partitioning</th>
<th>Inverse Table</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 4 6 8 10</td>
<td>2 4 6 8 10</td>
</tr>
<tr>
<td>Matrix Inverse</td>
<td>2 4 6 8 10</td>
<td>2 4 6 8 10</td>
</tr>
<tr>
<td>Matrix × Matrix</td>
<td>6 60 210 504 990</td>
<td>2 20 70 168 330</td>
</tr>
<tr>
<td>Matrix × Vector</td>
<td>4 16 36 64 100</td>
<td>4 16 36 64 100</td>
</tr>
<tr>
<td>Matrix + Matrix</td>
<td>2 36 150 392 810</td>
<td>1 14 55 140 285</td>
</tr>
<tr>
<td>Vector + Vector</td>
<td>2 12 30 56 90</td>
<td>2 12 30 56 90</td>
</tr>
</tbody>
</table>

It should be noted that in the compound matrix inversion method the full inverse matrix is available and the solution for further right-hand sides involves only additional matrix times vector operations. An exactly similar saving applies in the use of the factorized inverse table method which therefore retains its advantage in respect of the number of operations, but additional storage is required for the intermediate stages in this case. However, new solutions for modifications of the original problem parameters, affecting the coefficient matrix, are more conveniently accomplished by matrix inversion than by the factorized inverse table.

Iterative methods

The direct solution of large numbers of simultaneous linear equations involves many time-consuming operations and the handling of large numbers of coefficients. Using orthodox methods (i.e., all methods other than the diakoptic ones to be described in chapters 5–8), the solution of \( n \) linear equations involves the storage of \( n \times n \) coefficients. The time involved in the process of numerical inversion increases approximately as the cube of the number of equations.

There are many reasons why direct methods should be preferred but for matrices containing a large proportion of zero elements, such as arise in the solution of electrical networks, approximate methods which require only the coefficients of the original equations could involve less computer time and smaller storage.

There are many different systematic approximation methods
available and these are described in books on numerical analysis. One of the simplest and most effective methods, which is described in this chapter, is known as the Gauss-Seidel or Liebmann method [2].

**Gauss-Seidel iteration**

The method depends on the use of approximate values of all but one variable in order to obtain a better value of the remaining variable.

Consider the following set of equations:

\[
A_{11}X_1 + A_{12}X_2 + A_{13}X_3 = B_1
\]

\[
A_{21}X_1 + A_{22}X_2 + A_{23}X_3 = B_2
\]

\[
A_{31}X_1 + A_{32}X_2 + A_{33}X_3 = B_3
\]

which may be written as

\[
X_1 + \frac{A_{12}}{A_{11}}X_2 + \frac{A_{13}}{A_{11}}X_3 - \frac{B_1}{A_{11}} = 0
\]

\[
\frac{A_{21}}{A_{22}}X_1 + X_2 + \frac{A_{23}}{A_{22}}X_3 - \frac{B_2}{A_{22}} = 0
\]

\[
\frac{A_{31}}{A_{33}}X_1 + \frac{A_{32}}{A_{33}}X_2 + X_3 - \frac{B_3}{A_{33}} = 0
\]

or simply as

\[
X_1 + a_{12}X_2 + a_{13}X_3 - b_1 = 0
\]

\[
a_{21}X_1 + X_2 + a_{23}X_3 - b_2 = 0
\]

\[
a_{31}X_1 + a_{32}X_2 + X_3 - b_3 = 0
\]

where

\[
a_{12} = \frac{A_{12}}{A_{11}}, \quad a_{13} = \frac{A_{13}}{A_{11}}, \quad b_1 = \frac{B_1}{A_{11}} \quad \text{etc.}
\]

Denoting the n-th approximation to the solution of \( X \) by \( X^{(n)} \), the \( X^{(n+1)} \) solution can be obtained from \( X^{(n)} \) according to

\[
\begin{align*}
X_1^{(n+1)} &= b_1 - a_{12}X_2^{(n)} - a_{13}X_3^{(n)} \\
X_2^{(n+1)} &= b_2 - a_{21}X_1^{(n+1)} - a_{23}X_3^{(n)} \\
X_3^{(n+1)} &= b_3 - a_{31}X_1^{(n+1)} - a_{32}X_2^{(n+1)}
\end{align*}
\]

(3.7)
The values of $X^{(n+1)}$ replace those of $X^{(n)}$ in the iteration as soon as they have been computed. This is often referred to as updating the value of the variable.

It is evident that before proceeding with the calculations, some starting values for all but one of the unknowns must be assumed. The above process can be regarded as one of relaxation [13]. For each equation a residual $R_i^{(n)}$ ($i = 1, 2, 3$) after $n$ approximations is defined by

$$
\begin{align*}
X_1^{(n)} + a_{12} X_2^{(n)} + a_{13} X_3^{(n)} - b_1 &= R_1^{(n)} \\
 a_{21} X_1^{(n+1)} + X_2^{(n)} + a_{23} X_3^{(n)} - b_2 &= R_2^{(n)} \\
a_{31} X_1^{(n+1)} + a_{32} X_2^{(n+1)} + X_3^{(n)} - b_3 &= R_3^{(n)}
\end{align*}
$$

(3.8)

where $X_i^{(n)}$ ($i = 1, 2, 3$) is the $n$-th approximation to the unknown $X$. Since these values are approximate, in general the residuals will not be zero. The object of the Gauss-Seidel process is to make changes in the $X_i^{(n)}$ variable so as to reduce the residual $R_i$ to zero. By equating $R_i^{(n)}$ to zero in equations (3.8), these equations reduce to the original set as given by equations (3.7).

The above equations can also be written as

$$
\begin{align*}
X_1^{(n+1)} &= X_1^{(n)} + (b_1 - a_{12} X_2^{(n)} - a_{13} X_3^{(n)} - X_1^{(n)}) \\
 &= X_1^{(n)} - R_1^{(n)} \\
X_2^{(n+1)} &= X_2^{(n)} + (b_2 - a_{21} X_1^{(n+1)} - a_{23} X_3^{(n)} - X_2^{(n)}) \\
 &= X_2^{(n)} - R_2^{(n)} \\
X_3^{(n+1)} &= X_3^{(n)} + (b_3 - a_{31} X_1^{(n+1)} - a_{32} X_2^{(n)} - X_3^{(n)}) \\
 &= X_3^{(n)} - R_3^{(n)}
\end{align*}
$$

(3.9)

In many cases a considerable improvement of convergence can be obtained if in equations (3.9) the residuals are multiplied by an acceleration factor $K$ where $1 < K < 2$. The equations for the process then take the following forms:

$$
\begin{align*}
X_1^{(n+1)} &= X_1^{(n)} - KR_1^{(n)} \\
 &= X_1^{(n)} + K(b_1 - a_{12} X_2^{(n)} - a_{13} X_3^{(n)} - X_1^{(n)}) \\
X_2^{(n+1)} &= X_2^{(n)} - KR_2^{(n)} \\
 &= X_2^{(n)} + K(b_2 - a_{21} X_1^{(n+1)} - a_{23} X_3^{(n)} - X_2^{(n)})
\end{align*}
$$
Solution of Network Equations

\[ X_3^{(n+1)} = X_3^{(n)} - KR_3^{(n)} \]
\[ = X_3^{(n)} + K(b_3 - a_{31}X_1^{(n+1)} - a_{32}X_2^{(n+1)} - X_3^{(n)}) \]

**Numerical example**

As an example consider the numerical solution of the equations

\[ \begin{align*}
10X_1 - 2X_2 + X_3 &= 185 \\
-3X_1 - 6X_2 + 2X_3 &= 93 \\
X_1 - 2X_2 - 5X_3 &= 49
\end{align*} \]

\[ \begin{align*}
X_1 &= 0 \cdot 2X_2 - 0 \cdot 1X_3 + 18 \cdot 5 \\
X_2 &= -0 \cdot 5X_1 + 0 \cdot 333X_3 - 15 \cdot 5 \\
X_3 &= 0 \cdot 2X_1 - 0 \cdot 4X_2 - 9 \cdot 8
\end{align*} \]

Assuming \( X_2 = 0 \), and \( X_3 = 0 \) in the first equation, \( X_1 = 18 \cdot 5 \).

Using this value of \( X_1 \), and assuming \( X_3 = 0 \), \( X_2 \) can be found from the second equation, thus

\[ X_2 = -9 \cdot 25 + 0 - 15 \cdot 5 = -24 \cdot 75 \]

Using the values of \( X_1 \) and \( X_2 \) calculated above, the value of \( X_3 \) can be found from the third equation, thus

\[ X_3 = 3 \cdot 7 + 9 \cdot 9 - 9 \cdot 8 = 3 \cdot 8 \]

The process is repeated using the new values of \( X_2 \) and \( X_3 \) to get a new value of \( X_1 \) and so on to get columns \( X^{(2)} \), \( X^{(3)} \) ... \( X^{(6)} \)

\[
\begin{align*}
X^{(1)} &\quad X^{(2)} &\quad X^{(3)} &\quad X^{(4)} &\quad X^{(5)} &\quad X^{(6)} \\
18 \cdot 5 &\quad 13 \cdot 2 &\quad 14 \cdot 22 &\quad 13 \cdot 86 &\quad 13 \cdot 974 &\quad 13 \cdot 939 \\
-24 \cdot 7 &\quad -20 \cdot 8 &\quad -22 \cdot 22 &\quad -21 \cdot 79 &\quad -21 \cdot 925 &\quad -21 \cdot 882 \\
3 \cdot 8 &\quad 1 \cdot 2 &\quad 1 \cdot 93 &\quad 1 \cdot 69 &\quad 1 \cdot 765 &\quad 1 \cdot 741
\end{align*}
\]

**Convergence**

Convergence of the solution may be judged by comparing absolute value of the differences between the unknowns from two successive iterations. When this difference for each variable lies within a specified tolerance, the process can be terminated.

Not all types of equations can be solved by iterative methods
and for some the sequence of approximations has to be repeated many times. Convergence is rapid when the absolute values of the coefficients on the leading diagonal are large compared with the non-diagonal elements. This can readily be seen from the form of the equations for \(X_1^{(n+1)}, X_2^{(n+1)}\) ...

It is sometimes possible to accelerate convergence by applying factors during the process of iteration.

However, it is itself a difficult mathematical problem to determine the optimum acceleration factors [14] as these depend on the nature of the equations and empirical values are usually employed. For some equations a correct acceleration factor can reduce the time to convergence by a factor from two to fifty or more. Acceleration factors which are too large can cause divergence particularly during the early stages of calculations when there may be relatively large differences between successive approximations. Other methods of acceleration, such as Aitken’s method [4], have also been used.

**Ill-conditioned equations**

If the determinant \(\Delta\) of the coefficient matrix is small compared with some of the terms in its expression, then the solution of the equations is very sensitive to round-off errors in computation.

As an extreme case of this, consider the following example:

\[
1000X_1 + 2001X_2 = 4003
\]

\[
X_1 + 2X_2 = 4
\]

for which

\[
\Delta = -1
\]

and the true solution is

\[
X_1 = -2
\]

\[
X_2 = 3
\]

If the coefficient of \(X_2\) in the first equation is changed by \(-0.1\%\) to 1999, the solution becomes

\[
X_1 = 10
\]

\[
X_2 = -3
\]

If the same coefficient is changed by \(0.1\%\) to 2003 the solution becomes

\[
X_1 = 2
\]

\[
X_2 = 1
\]
If the coefficients in such a set of equations are subject to some uncertainty, either due to experimental errors or to rounding-off errors in preliminary calculations, then in such an extreme case as above, the solutions have no real meaning.

The numerical determination of the measure of ill-conditioning is a long process and it is often simpler in practice to recognize the effect by symptoms in the course of solution. If some of the coefficients on the leading diagonal are numerically small compared with some non-diagonal terms, it can suggest difficulty in solution. In general, ill-conditioned equations are more difficult to solve by iterative techniques than by direct methods.

Ill-conditioned equations may be due to the physical nature of the problem which the equations describe. If the problem is such that a small change in one of the parameters will result in a large actual change in the behaviour of the system, then there is very little that can be done with the equations. Care can be taken to ensure that the largest elements are taken as pivots and numerical calculations carried out with sufficient digits to minimize rounding errors. In such circumstances the solution should be carefully checked for accuracy.

If the physical problem is such that a small change in one of the parameters of the problem does not produce undue change in the behaviour of the problem and yet the equations which describe the problem display ill-conditioning effects, then it should be possible to re-formulate the equations so as to improve the conditioning. For example, if a mesh current method has been used without success then the nodal voltage method might be tried. Different ways of formulating and solving equations of a given problem are considered in detail in chapter 5.

Hybrid methods

Direct methods give a solution after a definite number of operations. The number of calculations increases approximately as the cube of the number of equations and the storage required for the coefficients is proportional to the square. Iterative methods require only the storage for the coefficients of the original equations but the amount of calculation depends upon the accuracy required and the nature of the problem. For some problems an iterative process can be very lengthy.

Many electrical problems are such that a group of nodes may be interconnected by many branches, but with relatively few
branches between similar groups. The coefficient matrices of these networks are characterized by groups of non-zero elements separated from other similar groups by many zero elements. Such problems may often be solved by a combination of direct and iterative methods so as to take advantage of this characteristic.

\[ I_a \quad Y_1 \quad I_b \quad Y_7 \quad I_c \quad Y_4 \quad I_d \]
\[ V_a \quad Y_2 \quad V_b \quad Y_3 \quad V_c \quad Y_5 \quad V_d \quad Y_6 \]

(a) original network

\[ I_a \quad Y_1 \quad i \quad Y_4 \quad I_d \]
\[ V_a \quad Y_2 \quad V_b \quad Y_3 \quad V_c \quad Y_5 \quad V_d \quad Y_6 \]

(b) equivalent network

Fig. 3.3

Consider as an example the network shown in fig. 3.3(a). The centre branch, \( Y_7 \), can be replaced by an equivalent current expressed in terms of the branch admittance and the voltage difference across the branch. Denoting the new variable by the letter \( i \),

\[ i = Y_7 (v_c - v_b) \]
The original network can now be replaced by an equivalent network as shown in fig. 3.3 (b).

Writing the equations for each part of the equivalent network separately gives

\[(Y_1 + Y_2) v_a - Y_1 v_b = I_a\]
\[-Y_1 v_a + (Y_1 + Y_3) v_b = I_b + i\]

and

\[(Y_4 + Y_5) v_c - Y_4 v_d = I_c - i\]
\[-Y_4 v_c + (Y_4 + Y_6) v_d = I_d\]

But

\[i = Y_7(v_c - v_b)\]

Substituting for \(i\) in the above equations

\[(Y_1 + Y_2) v_a - Y_1 v_b = I_a\]
\[-Y_1 v_a + (Y_1 + Y_3) v_b = I_b + Y_7 (v_c - v_b)\]

and

\[(Y_4 + Y_5) v_c - Y_4 v_d = I_c - Y_7 (v_c - v_b)\]
\[-Y_4 v_c + (Y_4 + Y_6) v_d = I_d\]

or

\[(Y_1 + Y_2) v_a - Y_1 v_b = I_a\]
\[-Y_1 v_a + (Y_1 + Y_3 + Y_7) v_b = I_b + Y_7 v_c\]

and

\[(Y_4 + Y_5 + Y_7) v_c - Y_4 v_d = I_c + Y_7 v_b\]
\[-Y_4 v_c + (Y_4 + Y_6) v_d = I_d\]

Writing these equations in matrix form gives

\[
\begin{bmatrix}
Y_{aa} & Y_{ab} \\
Y_{ba} & Y_{bb}
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b
\end{bmatrix}
= \begin{bmatrix}
I_a + 0 \\
I_b + Y_7 v_c
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
Y_{cc} & Y_{cd} \\
Y_{dc} & Y_{dd}
\end{bmatrix}
\begin{bmatrix}
v_c \\
v_d
\end{bmatrix}
= \begin{bmatrix}
I_c + Y_7 v_b \\
I_d + 0
\end{bmatrix}
\]

Note that the elements of the individual admittance matrices
are the same as the corresponding elements of the original network and these equations could have been written by inspection of the original network.

The solution can now be obtained by inverting the individual admittance matrices

\[
\begin{bmatrix}
    v_a \\
v_b
\end{bmatrix} = \begin{bmatrix}
    Z_{aa} & Z_{ab} \\
    Z_{ba} & Z_{bb}
\end{bmatrix} \begin{bmatrix}
    I_a + 0 \\
    I_b + Y_7 v_c
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    v_c \\
v_d
\end{bmatrix} = \begin{bmatrix}
    Z_{cc} & Z_{cd} \\
    Z_{dc} & Z_{dd}
\end{bmatrix} \begin{bmatrix}
    I_c + Y_7 v_b \\
    I_d + 0
\end{bmatrix}
\]

Denoting the \( n \)-th approximation to the solution of \( v \) by \( v^{(n)} \), the solution of the particular problem can be obtained by repeating the following sequence:

\[
\begin{align*}
    v_a^{(n+1)} &= Z_{aa} I_a + Z_{ab} I_b + Z_{ab} Y_7 v_c^{(n)} \\
    v_b^{(n+1)} &= Z_{ba} I_a + Z_{bb} I_b + Z_{bb} Y_7 v_c^{(n)} \\
    v_c^{(n+1)} &= Z_{cc} I_c + Z_{cd} I_d + Z_{cc} Y_7 v_b^{(n+1)} \\
    v_d^{(n+1)} &= Z_{dc} I_c + Z_{dd} I_d + Z_{dc} Y_7 v_b^{(n+1)}
\end{align*}
\]

Before proceeding with this sequence, some starting value for \( v_c^{(n)} \) must be assumed.

The hybrid method is an application of the Gauss-Seidel method using matrices instead of single elements.

In general, any network can be divided into a number of sub-networks. When every interconnection between nodes is represented by an equivalent current, the sequence is identical to the Gauss-Seidel method.

This concept of taking advantage of the physical nature of the network with division followed by inversion of the coefficient sub-matrices is the basis of 'tearing' or 'diakoptics' on which the later chapters are built. However, diakoptical methods are not restricted to iterative techniques.

**Numerical example**

To illustrate the hybrid method, consider the network shown in fig. 3.4(a).
The admittance matrices of the equivalent network shown in fig. 3.4 (b) together with the admittance of the removed branch can be formulated by inspection as

\[
Y_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 2.1 \end{bmatrix} \quad \text{and} \quad Y_{22} = \begin{bmatrix} 4.1 & -2 \\ -4 & 4 \end{bmatrix}
\]

Inverting the admittance submatrices

\[
Z_{11} = \begin{bmatrix} 0.656 & 0.313 \\ 0.313 & 0.625 \end{bmatrix} \quad \text{and} \quad Z_{22} = \begin{bmatrix} 0.323 & 0.161 \\ 0.161 & 0.331 \end{bmatrix}
\]

the equations of solution are
\[
\begin{bmatrix}
  v_a \\
  v_b
\end{bmatrix} = \begin{bmatrix}
  0.656 & 0.313 \\
  0.313 & 0.625
\end{bmatrix} \begin{bmatrix}
  1.5 + 0 \\
  2 + 0.1 v_c
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  v_c \\
  v_d
\end{bmatrix} = \begin{bmatrix}
  0.323 & 0.161 \\
  0.161 & 0.331
\end{bmatrix} \begin{bmatrix}
  0.5 + 0.1 v_b \\
  2.5 + 0
\end{bmatrix}
\]

Multiplying out, the iterative equations are

\[
v_a^{(n+1)} = (0.656)(1.5) + (0.313)(2) + (0.313)(0.1)v_c^{(n)}
\]
\[
= 1.609 + 0.031 v_c^{(n)}
\]

\[
v_b^{(n+1)} = (0.313)(1.5) + (0.625)(2) + (0.625)(0.1)v_c^{(n)}
\]
\[
= 1.719 + 0.063 v_c^{(n)}
\]

\[
v_c^{(n+1)} = (0.323)(0.5) + (0.161)(2.5) + (0.323)(0.1)v_b^{(n+1)}
\]
\[
= 0.565 + 0.032 v_b^{(n)}
\]

\[
v_d^{(n+1)} = (0.161)(0.5) + (0.331)(2.5) + (0.161)(0.1)v_b^{(n+1)}
\]
\[
= 0.907 + 0.033 v_b^{(n+1)}
\]

Taking the initial value of \( v_c^{(0)} \) to be zero and by working with a digital computer to nine digits and quoting to five decimal digits the following sequence of approximations is obtained:

<table>
<thead>
<tr>
<th>( v^{(1)} )</th>
<th>( v^{(2)} )</th>
<th>( v^{(3)} )</th>
<th>( v^{(4)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.60937</td>
<td>1.62875</td>
<td>1.62879</td>
<td>1.62879</td>
</tr>
<tr>
<td>1.71875</td>
<td>1.75750</td>
<td>1.75758</td>
<td>1.75758</td>
</tr>
<tr>
<td>0.61996</td>
<td>0.62121</td>
<td>0.62121</td>
<td>0.62121</td>
</tr>
<tr>
<td>0.93498</td>
<td>0.93560</td>
<td>0.93561</td>
<td>0.93561</td>
</tr>
</tbody>
</table>

Solving the original problem by the Gauss-Seidel iterative method, taking the initial value of \( v^{(0)} \) to be zero, results in the following sequence of approximations:

<table>
<thead>
<tr>
<th>( v^{(1)} )</th>
<th>( v^{(2)} )</th>
<th>( v^{(3)} )</th>
<th>( v^{(4)} )</th>
<th>( v^{(5)} )</th>
<th>( v^{(6)} )</th>
<th>( v^{(7)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75000</td>
<td>1.40476</td>
<td>1.56432</td>
<td>1.62869</td>
<td>1.62876</td>
<td>1.62878</td>
<td>1.62879</td>
</tr>
<tr>
<td>1.30952</td>
<td>1.62864</td>
<td>1.72130</td>
<td>1.75752</td>
<td>1.75756</td>
<td>1.75757</td>
<td>1.75757</td>
</tr>
<tr>
<td>0.15389</td>
<td>0.50409</td>
<td>0.59176</td>
<td>0.62118</td>
<td>0.62120</td>
<td>0.62121</td>
<td>0.62121</td>
</tr>
<tr>
<td>0.70195</td>
<td>0.87704</td>
<td>0.92088</td>
<td>0.93559</td>
<td>0.93560</td>
<td>0.93561</td>
<td>0.93561</td>
</tr>
</tbody>
</table>
It is seen from these results that five decimal digits accuracy is achieved after only three cycles using the Hybrid method, whereas eleven cycles are required using the Gauss-Seidel method. In comparing the total number of arithmetic operations involved, account must be taken of the matrix inversion process required for the hybrid method. This will vary, depending on the size and nature of the problem and the manner of subdivision.

Choice of method

With every organized digital computer service standard programmes will exist, at least for the inversion of real matrices. However, if a choice is available or if there are sufficient cases to justify the development of a special method, there are a number of factors to be considered. Among these the most important are:

1. The method of carrying out the calculation, i.e. slide rule, desk machine, or digital computer.
2. The number of equations and available storage space in the digital computer.
3. The form of the equations, i.e. symmetrical, diagonal, containing many zero elements, etc.
4. The type of equations, i.e. well- or ill-conditioned.
5. The form of solution required, i.e. for all unknowns, a few unknowns, or perhaps even only one unknown.
6. The number of times a similar problem is likely to occur.
7. Accuracy: All methods should contain final checks, preferably by re-substitution in the original equations.
8. Whether modified solutions are required either in respect of changed driving functions (right-hand sides) or changed physical parameters affecting the coefficient matrix.

Comparison of available methods

Determinants. Because of the considerable number of arithmetic operations involved, these should be used only for equations with two or three unknowns.

Direct. There are many methods, each differing in the manner of inverting the matrix, but in all the methods the results can be obtained directly from the inverted matrix. When the time of
calculation is no problem and the necessary storage space is available, direct methods should be preferred.

**Iterative.** These methods can be powerful methods for certain types of equations, in particular those characterized by coefficient matrices containing a large proportion of zero elements. They are not very suitable for ill-conditioned equations and they can be additionally time-consuming when a number of solutions are required for equations having the same coefficients.

**Hybrid.** These methods are useful for large numbers of equations containing a large proportion of zero elements and when a number of solutions are required for different driving functions. More suitable to ill-conditioned equations than an iterative method alone, they can be used to considerable advantage in many electrical network problems arising in power system analysis.

**Problems**

1. Solve, by successive elimination and Gauss-Seidel iterative method, the set of equations:

   (a) \[ 8X + 2Y + 3Z = 9 \]
   \[ X + 7Y - 3Z = 4 \]
   \[ 2X - 3Y + 9Z = 1 \]

   (b) \[ 3.421X_1 + 1.234X_2 + 0.736X_3 + 0.124X_4 = 0.365 \]
   \[ 1.202X_1 + 3.575X_2 + 0.874X_3 + 0.210X_4 = 0.256 \]
   \[ 0.422X_1 + 0.543X_2 + 3.428X_3 + 0.428X_4 = 0.444 \]
   \[ 0.116X_1 + 0.256X_2 + 0.488X_3 + 3.627X_4 = 0.868 \]

2. By eliminating \(X_3, X_4\) and \(X_5\) in one step, solve the following sets of equations directly for \(X_1, X_2\) and therefore \(X_3, X_4, X_5\):

   (a) \[ X_1 + 2X_2 - 3X_3 + 4X_4 + 5X_5 = 10 \]
   \[ 2X_1 + 4X_2 + 3X_3 + 5X_4 - X_5 = 9 \]
   \[ 3X_1 + 4X_2 + 5X_3 + 2X_4 + 3X_5 = 8 \]
   \[ X_1 + 2X_2 - 4X_3 - 3X_4 + 5X_5 = 7 \]
   \[ 5X_1 + X_2 - 3X_3 + 3X_4 + 2X_5 = 6 \]
Solution of Network Equations

(b) \[ 3X_1 + 4X_2 - 2X_3 + 6X_4 - X_5 = 2 \]
\[ 2X_2 + 7X_4 = 0 \]
\[ X_1 - X_2 + X_3 - X_5 = -3 \]
\[ X_1 + X_3 + 3X_4 = 0 \]
\[ 3X_2 - X_3 + 2X_5 = 1 \]

3. Verify the results of the problems given in exercises 1 and 2 by solving the equations using (a) matrix partitioning, and (b) the factorized inverse table.

---

**Fig. 3.5**

**Fig. 3.6**
4. The equations describing the behaviour of the networks shown in fig. 3.5 and fig. 3.6 are

(a) \[
\begin{bmatrix}
3 & -1 & 0 & -1 \\
-1 & 3 & -1 & 0 \\
0 & -1 & 3 & -1 \\
-1 & 0 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d
\end{bmatrix}
= \begin{bmatrix} 3 \\ 33 \\ 0 \\ 0 \end{bmatrix}
\]

(b) \[
\begin{bmatrix}
3 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 3 & -1 & -1 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & -1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e
\end{bmatrix}
= \begin{bmatrix} -4 \\ 5 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\]

Find corresponding equations which describe the behaviour of the same networks reduced to nodes a and b.

5. Using the hybrid method determine the nodal voltages for the networks shown in (a) fig. 3.5 and (b) fig. 3.6.
CHAPTER FOUR
Linear Transformations

Introduction
Modern engineering practice involves the analysis of large and complex problems. Much vital work in this field can be accomplished by the direct application of matrix algebra alone. In present-day engineering practice, problems are becoming more complex in character and the formulation of the equations more difficult. Even when the problem is capable of being expressed in equation form, the computation is often lengthy when using ordinary matrix algebra.

The recent advent of high-speed digital computers, with large core storage, has made it practicable to solve by direct numerical methods large varieties of physical problems. The application of digital computers to such problems requires the mathematical formulation of the problem to be accomplished in a logical and organized manner.

Kron [10] is the author of a method of analysis of electrical networks based on tensors. This analysis provides routine procedures by which large and complex systems can be transformed to related systems whose analysis is simpler.

The basis of Kron's method of analysis is a system of linear transformations which is used to assemble the functional equations of a complex system from a knowledge of the equations of component parts and the way in which these parts are interconnected. Tensor notation is well suited to expressing such transformations for general analysis, but, as is shown in this chapter, a great deal can be accomplished in electrical network analysis by the use of matrix algebra alone without recourse to tensor analysis.

Mesh current analysis
A well-known example of the use of a linear transformation is the solution of an electrical network by the mesh current method in which the mathematical solution is often simplified by introducing hypothetical mesh currents. The values of the branch currents can be found from the values of the mesh currents and the linear relationships between them. The manipulation of the equations in this
manner is known as a linear transformation between the mesh and branch methods of defining the network.

For simple networks the equations based on mesh current analysis can be written down directly by inspection. For more complicated networks, matrix algebra can be used to assemble the mesh equations from the branch equations (of the form \( Z_i = E \)). The method which is described below serves to illustrate the general procedure of establishing simple relations in matrix form and the routine steps used to assemble the functional equations of a complex system from the simple equations.

![Diagram](attachment:image.png)

(a) original network

![Diagram](attachment:image.png)

(b) primitive network

Fig. 4.1

To illustrate the method consider the network shown in fig. 4.1 (a). For simplicity assume that there is no mutual inductance between any of the branch impedances. In chapter 2 it has been shown that the equations which describe the performance of the network can be set up as
\[
\begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & 0 \\
-Z_4 & (Z_2 + Z_4 + Z_5) & -Z_5 \\
0 & -Z_5 & (Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix} =
\begin{bmatrix}
E_p \\
0 \\
-E_r
\end{bmatrix}
\]

Using general terms and subscripts, the above equation can be written in concise form as
\[
\mathbf{Z}_{\beta \beta} \mathbf{i}_{\beta} = \mathbf{E}_{\beta}
\]

(4.1)

**Primitive network**

Consider now the network shown in fig. 4.1(b). This network is constructed by breaking up the original network into individual branch components. Each individual branch impedance is supplied with a hypothetical voltage source which gives rise to the same current through the branch as in the original network. The component network is called the primitive network.

In the primitive network the current flowing in branch \( Z_j \) is \( i_j \) (j = 1, 2, ..., 5) which is the same as in branch \( Z_j \) of the original network. It follows therefore that the hypothetical voltage source \( e_j \) must be numerically equal to the voltage \( v_j \).

The equation describing the performance of the primitive network can be established very simply by Ohm's law as
\[
\begin{align*}
Z_1 i_1 &= e_1 \\
Z_2 i_2 &= e_2 \\
Z_3 i_3 &= e_3 \\
Z_4 i_4 &= e_4 \\
Z_5 i_5 &= e_5
\end{align*}
\]

The above equations may be written in matrix form as
\[
\begin{bmatrix}
Z_1 & \cdots & \cdots \\
\vdots & Z_2 & \cdots \\
\vdots & \vdots & Z_3 \\
\vdots & \vdots & \vdots & Z_4 \\
\vdots & \vdots & \vdots & \vdots & Z_5
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5
\end{bmatrix} =
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{bmatrix}
\]
The dots indicate zero elements.

This matrix equation may be written more concisely in general terms, using subscripts, as

\[ Z_{\psi \psi} i_\psi = e_\psi \]  

(4.2)

The branch impedance matrix \( Z_{\psi \psi} \) can be established very simply by inspection. It consists of all branch impedance elements listed on the diagonal and all other elements of the matrix are zero.

The mesh impedance matrix \( Z_{\beta \beta} \) of the original network can be obtained from the branch impedance matrix \( Z_{\psi \psi} \) of the primitive network by a routine procedure to be described in the following two sections. Once the matrix \( Z_{\beta \beta} \) is established, the equations of the original network \( Z_{\beta \beta} i_\beta = E_\beta \) can be constructed by completing the vectors \( i_\beta \) and \( E_\beta \) by inspection.

Connection matrix

Using Kirchhoff's law the currents in the primitive network may be expressed in terms of the mesh currents defined in the original network. From fig. 4.1(a)

\[
\begin{align*}
i_1 &= i_p \\
i_2 &= i_q \\
i_3 &= i_r \\
i_4 &= i_p - i_q \\
i_5 &= i_q - i_r
\end{align*}
\]

The above relations may be written in matrix form as

\[
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5
\end{bmatrix} =
\begin{bmatrix}
1 & . & . \\
. & 1 & . \\
. & . & 1 \\
1 & -1 & . \\
. & 1 & -1
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix}
\]

or in general terms as

\[ i_\psi = C_{\psi \beta} i_\beta \]  

(4.3)
Since there are no mutual impedances in the network, the positive direction of current flowing through the branches may be assumed arbitrarily. If the positive direction of current is changed, then the corresponding elements of the coefficient matrix \( C_{\psi\beta} \) will change sign.

In order to define the elements of the connection matrix \( C_{\psi\beta} \), it is desirable to attribute direction to a branch as distinct from the current in or voltage across the branch. The direction of a branch is assigned arbitrarily and is distinct from the arbitrary direction of the current through the branch. It is, however, convenient to associate the branch direction with the direction of the current through the branch and since both directions are assigned arbitrarily, for the purpose of this book they are assumed to coincide.

Using the above assumption, the element \( C_{jp} \) in the row \( j \) and column \( p \) of the connection matrix \( C_{\psi\beta} \), corresponding to the branch \( j \) and mesh \( p \), may be defined as

\[
C_{jp} = \begin{cases} 
+1, & \text{if the direction of the mesh } p \text{ coincides with the direction of the branch } j. \\
-1, & \text{if the direction of the mesh } p \text{ is opposite to the direction of the branch } j. \\
0, & \text{if the mesh } p \text{ does not include the branch } j.
\end{cases}
\]

The matrix \( C_{\psi\beta} \) is known as the connection matrix or mesh incidence matrix. It consists of as many rows as there are branches in the network and as many columns as there are independent mesh currents.

Using the definition, the elements of the connection matrix can be established by inspection. Since the elements of the connection matrix \( C_{\psi\beta} \) consist of \(+1, -1\) or \(0\), multiplication by \( C_{\psi\beta} \) reduces to simple summation only. Details of this are given in chapter 7. It is the connection matrix which defines the transformation between the (hypothetical) mesh currents and the branch currents as a linear transformation.

**Transformation law**

The transformation law relating the impedance matrix \( Z_{\beta\beta} \) of the original network to the impedance matrix \( Z_{\psi\psi} \) of the primitive
network may be developed using the principle of power conservation or by a topological method. In order to show the concepts involved, both methods are developed in the following two sections.

*Power conservation.* The transformation law may be developed using the equations derived in the previous section.

\[
\begin{align*}
Z_{\beta\beta} i_{\beta} & = E_{\beta} & \quad (4.1) \\
Z_{\psi\psi} i_{\psi} & = e_{\psi} & \quad (4.2) \\
i_{\psi} & = C_{\psi\beta} i_{\beta} & \quad (4.3)
\end{align*}
\]

Substituting for \( i_{\psi} \) from equation (4.3) into equation (4.2) results in

\[
Z_{\psi\psi} C_{\psi\beta} i_{\beta} = e_{\psi} \quad (4.4)
\]

An additional equation is required to complete the transformation and this may be established from the following consideration. The total power supplied to the network from the voltage sources may be expressed, as shown in chapter 2, as \( E_{\beta}^t i_{\beta}^* \).

The total power dissipated by the network may be expressed as \( e_{\psi}^t i_{\psi}^* \), where \( t \) indicates matrix transposition and \( * \) indicates complex conjugate.

Since there are no other sources or sinks of power then, by conservation of power,

\[
e_{\psi}^t i_{\psi}^* = E_{\beta}^t i_{\beta}^*
\]

Substituting for \( i_{\psi}^* \) in the above equation from equation (4.3),

\[
e_{\psi}^t C_{\psi\beta}^* i_{\beta}^* = E_{\beta}^t i_{\beta}^*
\]

Since \( i_{\beta}^* \) is an arbitrary vector and can take any value, it follows from chapter 1 that

\[
e_{\psi}^t C_{\psi\beta}^* = E_{\beta}^t
\]

Transposing,

\[
C_{\beta\psi}^* e_{\psi} = E_{\beta}
\]

This equation expresses the linear transformation between voltage sources in the meshes and voltages across the branches. It involves the connection matrix \( C_{\beta\psi}^* \) and it is complementary to equation (4.3).
Substituting for $e_\psi$ in the above equation from equation (4.4) results in

$$C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta} i_\beta = E_\beta$$

Comparing with equation (4.1),

$$Z_{\beta\beta} = C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta}$$  \hspace{1cm} (4.5)

Equation (4.5) transforms the branch impedance matrix $Z_{\psi\psi}$ to the mesh impedance matrix $Z_{\beta\beta}$ using the connection matrices $C_{\psi\beta}$ and $C_{\beta\psi}^t$ whose elements are $+1, -1$ or $0$. This is also a linear transformation.

*Topological method.* The transformation law may also be developed using the same initial equations as are used in the power conservation derivation, namely

$$Z_{\beta\beta} i_\beta = E_\beta$$  \hspace{1cm} (4.1)

$$Z_{\psi\psi} i_\psi = e_\psi$$  \hspace{1cm} (4.2)

$$i_\psi = C_{\psi\beta} i_\beta$$  \hspace{1cm} (4.3)

$$Z_{\psi\psi} C_{\psi\beta} i_\beta = e_\psi$$  \hspace{1cm} (4.4)

The additional equation, which replaces the conservation of power relationship, is obtained by expressing the voltage sources in the individual meshes in terms of the voltages across the branches in the respective meshes, using Kirchhoff’s law.

From fig. 4.1(a)

$$v_1 + v_4 = E_p$$

$$v_2 - v_4 + v_5 = E_q = 0$$

$$v_3 - v_5 = -E_r$$

The hypothetical voltage sources shown in fig. 4.1(b) are numerically equal to the voltages across the corresponding branches, that is

$$v_1 = e_1, \; v_2 = e_2, \; ..., \; v_5 = e_5$$

therefore

$$e_1 + e_4 = E_p$$

$$e_2 - e_4 + e_5 = E_q = 0$$

$$e_3 - e_5 = -E_r$$
The above relationships may be written in matrix form as

\[
\begin{bmatrix}
1 & \cdots & 1 & \cdots & . \\
. & 1 & \cdots & -1 & 1 \\
. & . & 1 & \cdots & -1
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{bmatrix}
= \begin{bmatrix}
E_p \\
0 \\
-E_r
\end{bmatrix}
\]

or in general terms as

\[\mathbf{B}_{\beta\psi} \mathbf{e}_\psi = \mathbf{E}_\beta\]  \hspace{1cm} (4.6)

The matrix \(\mathbf{B}_{\beta\psi}\) consists of as many rows as there are independent meshes and as many columns as there are branches in the network. The element \(B_{pj}\) in the row \(p\) and column \(j\) may be defined as

\[B_{pj} = \begin{cases} 
+1, & \text{if the direction of the branch } j \text{ coincides with the direction of the mesh } p. \\
-1, & \text{if the direction of the branch } j \text{ is opposite to the direction of the mesh } p. \\
0, & \text{if the branch } j \text{ is not in the mesh } p.
\end{cases}\]

It can be seen by comparing the definition of \(\mathbf{C}_{\psi\beta}\), shown on p. 87, with the definition for \(\mathbf{B}_{\beta\psi}\), that the two matrices are simply related as

\[\mathbf{B}_{\beta\psi} = (\mathbf{C}_{\beta\psi})^t\]

Therefore equation (4.6) may be expressed as

\[\mathbf{C}_{\beta\psi}^t \mathbf{e}_\psi = \mathbf{E}_\beta\]

Substituting in the above equation for \(\mathbf{e}_\psi\) from equation (4.4) results in

\[\mathbf{C}_{\beta\psi}^t \mathbf{Z}_{\psi\psi} \mathbf{C}_{\psi\beta} \mathbf{i}_\beta = \mathbf{E}_\beta\]

Comparing with equation (4.1)

\[\mathbf{Z}_{\beta\beta} = \mathbf{C}_{\beta\psi}^t \mathbf{Z}_{\psi\psi} \mathbf{C}_{\psi\beta}\]  \hspace{1cm} (4.7)

Although \(\mathbf{C}_{\beta\psi}^t\) appears in equation (4.5) and \(\mathbf{C}_{\beta\psi}\) appears in equation (4.7), it must be noted that for most applications considered in this book the elements of \(\mathbf{C}_{\psi\beta}\) are real numbers.
Linear Transformations

consisting of $+1, -1, 0$, therefore

$$C^*_{\psi \beta} = C_{\psi \beta}$$

and

$$C^t_{\beta \psi} = C^t_{\beta \psi}$$

hence equations (4.5) and (4.7) take identical form.

In certain applications, for example, networks including transformers with complex turn ratios, the elements of the connection matrix $C_{\psi \beta}$ may be complex numbers and the effective mesh current is not defined by topological consideration but must be derived from power invariance concepts. In such circumstances equation (4.5), derived from power invariance concepts, is applicable.

On the other hand, there are applications of the use of linear transformations in which the concept of power invariance is not valid. An example of this is shown in the section dealing with symmetrical components.

Example

Consider the network shown in fig. 4.1(a) and fig. 4.1(b). The impedance matrix $Z_{\psi \psi}$ and the connection matrix $C_{\psi \beta}$ can be constructed by inspection as

$$Z_{\psi \psi} = \begin{bmatrix} Z_1 & \cdots & \cdots & \cdots \\ \cdot & Z_2 & \cdots & \cdots \\ \cdot & \cdot & Z_3 & \cdots \\ \cdot & \cdot & \cdot & Z_4 \\ \cdot & \cdot & \cdot & \cdot & Z_5 \end{bmatrix}$$

$$C_{\psi \beta} = \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 2 & \cdot & 1 & \cdot \\ 3 & \cdot & \cdot & 1 \\ 4 & 1 & -1 & \cdot \\ 5 & \cdot & 1 & -1 \end{bmatrix}$$

The connection matrix $C_{\psi \beta}$ can be constructed by inspection.

To avoid mistakes in compiling the elements of the connection matrix it is helpful to number the rows of the matrix corresponding to the branch numbers ($1, 2, \ldots, 5$) and the columns corresponding to meshes ($p, q, r$).

Using equation (4.7) and performing the multiplication $C^t_{\beta \psi} Z_{\psi \psi}$
Multiplying \((C'_{\beta\psi} Z_{\psi\beta})\) results in the matrix \(Z_{\beta\beta}\) which is the mesh impedance coefficient matrix of the original network.

\[
\begin{bmatrix}
1 & 1 \\
1 & -1 & 1 \\
. & 1 & -1 \\
. & . & . & Z_5
\end{bmatrix}
\begin{bmatrix}
Z_1 & . & . & . \\
. & Z_2 & . & . \\
. & . & Z_3 & . \\
. & . & . & Z_4
\end{bmatrix} =
\begin{bmatrix}
Z_1 & . & . & Z_4 \\
. & Z_2 & . & -Z_4 \\
. & . & Z_3 & . \\
. & . & . & -Z_5
\end{bmatrix}
\]

In the above simple example the ordinary method of forming equations is faster since the matrix \(Z_{\beta\beta}\) can be written down directly and easily by inspection. As the complexity of the network increases, the use of connection matrices becomes more convenient. For example, the risk of errors in the formulation of equations of networks with cross-over branches is diminished and the difficulties presented by networks with mutual impedances are reduced by the use of connection matrices as shown in the following section. Also, as shown on p.100, equations for networks containing transformers with real or complex turns ratio can be established easily using connection matrices.

**Networks with mutual impedances**

The linear transformation method can be easily applied to networks with symmetrical or non-symmetrical mutual impedances.

Consider the network of fig. 4.2(a) with mutual impedances as shown. The polarity sense is indicated by the numbers 1 and 2 according to the convention given in chapter 2, that is, positive direction of current through the branches is from the end marked 1 into the end marked 2.
Following the steps given in the previous section, the primitive network is established as shown in fig. 4.2(b). The equations of the primitive network are set up as

\[ Z_1 i_1 + M_{12} i_2 + M_{14} i_4 = e_1 \]
\[ M_{21} i_1 + Z_2 i_2 + M_{23} i_3 = e_2 \]
\[ M_{32} i_2 + Z_3 i_3 = e_3 \]
\[ M_{41} i_1 + Z_4 i_4 = e_4 \]
\[ Z_5 i_5 = e_5 \]
The above equations may be written in matrix form as

\[
\begin{bmatrix}
Z_1 & M_{12} & \cdots & M_{14} & \cdots \\
M_{21} & Z_2 & \cdots & M_{23} & \cdots \\
\vdots & M_{32} & Z_3 & \cdots & \vdots \\
M_{41} & \cdots & Z_4 & \cdots & \vdots \\
\cdots & \cdots & \cdots & Z_5 & \cdots
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5
\end{bmatrix}
= 
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{bmatrix}
\]

or more concisely in general terms using subscripts as

\[Z_{\psi \psi} i_{\psi} = e_{\psi}\]

The impedance matrix \(Z_{\psi \psi}\) can be established very simply by inspection. It consists of all branch impedances listed on the diagonal elements and the mutual impedances listed on the non-diagonal elements. For instance, the mutual \(M_{14}\) between branches 1 and 4 is listed in the element of the first row and the fourth column and the mutual \(M_{32}\) between branches 3 and 2 listed in the element of the third row and second column. When there is no mutual impedance between two branches then the corresponding element in the matrix \(Z_{\psi \psi}\) is zero.

The impedance matrix \(Z_{\beta \beta}\), and hence the equations \(Z_{\beta \beta} i_{\beta} = E_{\beta}\) of the original network shown in fig. 4.2(a), can be established from the impedance matrix \(Z_{\psi \psi}\) of the primitive network, using equation (4.7)

\[Z_{\beta \beta} = C_{\psi \beta}^t \times Z_{\psi \psi} \times C_{\psi \beta}\]  \hspace{1cm} (4.7)

The connection matrix \(C_{\psi \beta}\) is obtained by expressing the currents in the primitive network in terms of the defined mesh currents in the original network fig. 4.2(a)

\[
i_1 = i_p \\
i_2 = i_q \\
i_3 = -i_r \\
i_4 = i_p - i_q \\
i_5 = i_q - i_r
\]
The above equations may be written in matrix form as

\[
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
\end{bmatrix} =
\begin{bmatrix}
1 & . & . \\
. & 1 & . \\
. & . & -1 \\
1 & -1 & . \\
. & 1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
\end{bmatrix}
\]

or in general terms as

\[
i_\psi = C_{\psi \beta} i_\beta
\]

Therefore

\[
C_{\psi \beta} =
\begin{bmatrix}
1 & . & . \\
. & 1 & . \\
. & . & -1 \\
1 & -1 & . \\
. & 1 & -1 \\
\end{bmatrix}
\]

Performing the multiplication \(C_{\beta \psi}^t Z_{\psi \psi}\),

\[
\begin{bmatrix}
1 & . & . & 1 & . \\
. & 1 & . & -1 & 1 \\
. & . & 1 & . & -1 \\
\end{bmatrix}
\begin{bmatrix}
Z_1 & M_{12} & . & M_{14} & . \\
M_{21} & Z_2 & M_{23} & . & . \\
. & M_{32} & Z_3 & . & . \\
M_{41} & . & . & Z_4 & . \\
. & . & . & . & Z_5 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(Z_1 + M_{41}) & M_{12} & . & (M_{14} + Z_4) & . \\
(M_{21} - M_{41}) & Z_2 & M_{23} & -Z_4 & Z_5 \\
. & -M_{32} - Z_3 & . & . & -Z_5 \\
\end{bmatrix}
\]

Multiplying \((C_{\beta \psi}^t Z_{\psi \psi}) C_{\psi \beta}\) results in the matrix \(Z_{\beta \beta}\) which is the mesh impedance matrix of the original network.
\[
\begin{bmatrix}
(Z_1 + M_{41}) & M_{12} & \cdots & (M_{14} + Z_4) & \cdots \\
(M_{21} - M_{41}) & Z_2 & M_{23} & -Z_4 & Z_5 \\
\cdots & -M_{32} - Z_3 & \cdots & -Z_5
\end{bmatrix}
\begin{bmatrix}
1 \\
\vdots \\
1 -1 \\
\cdots 1
\end{bmatrix}
\]

Interconnection of networks

In order to write the equations for large and complex networks it is not necessary to subdivide into individual branches. It is sufficient to subdivide into a number of smaller networks, to write down the equations for each individual network independently and then to transform them into the equations of the original network.

(a) original network

(b) primitive network

Fig. 4.3
Consider the network shown in fig. 4.3(a) subdivided into two parts. The primitive network is obtained by short circuiting each part through hypothetical voltage sources $e_2$ and $e_3$, as shown in fig. 4.3(b). In addition, voltage sources $E_p$ and $E_r$ appear in the primitive network as $e_1$ and $e_4$ respectively. The equations of the individual networks may be written by inspection as

\[
\begin{bmatrix}
(Z_1 + Z_4) & -Z_4 \\
-Z_4 & Z_4
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
= 
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
(Z_2 + Z_5) & -Z_5 \\
-Z_5 & (Z_3 + Z_6)
\end{bmatrix}
\begin{bmatrix}
i_3 \\
i_4
\end{bmatrix}
= 
\begin{bmatrix}
e_3 \\
e_4
\end{bmatrix}
\]

or in general terms as

\[Z_{\delta\delta} i_{\delta} = e_{\delta} \quad \text{and} \quad Z_{\theta\theta} i_{\theta} = e_{\theta}\]

Combining the two sets of equations into one compound matrix equation,

\[
\begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & 0 \\
-Z_4 & Z_4 & 0 \\
0 & (Z_2 + Z_5) & -Z_5 \\
0 & -Z_5 & (Z_3 + Z_6)
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4
\end{bmatrix}
= 
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4
\end{bmatrix}
\]

This may be written in general terms as

\[
\begin{bmatrix}
Z_{\delta\delta} : 0 \\
\cdots \\
0 : Z_{\theta\theta}
\end{bmatrix}
\begin{bmatrix}
i_{\delta} \\
\cdots \\
i_{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
e_{\delta} \\
\cdots \\
e_{\theta}
\end{bmatrix}
\]

and in a more concise form as

\[Z_{\psi\psi} i_{\psi} = e_{\psi}\]

The individual mesh currents $i_1$, $i_2$, $i_3$, $i_4$ of the primitive network are related to the mesh currents $i_p$, $i_q$, $i_r$ of the original network from fig. 4.3(a) and fig. 4.3(b) as
\[ i_1 = i_p \]
\[ i_2 = i_q \]
\[ i_3 = -i_q \]
\[ i_4 = -i_r \]

Since in the primitive network there are only four mesh currents, only four relationships are required.

The above equation in matrix form is
\[
\begin{bmatrix}
  i_1 \\
  i_2 \\
  \vdots \\
  i_3 \\
  i_4 
\end{bmatrix}
= 
\begin{bmatrix}
  1 & . & . \\
  . & 1 & . \\
  \vdots & \vdots & \vdots \\
  . & . & -1 
\end{bmatrix}
\begin{bmatrix}
  i_p \\
  i_q \\
  i_r 
\end{bmatrix}
\]

or in compound matrix form,
\[
\begin{bmatrix}
  i_{\delta} \\
  \vdots \\
  i_{\theta} 
\end{bmatrix}
= 
\begin{bmatrix}
  C_{\delta\beta} \\
  \vdots \\
  C_{\theta\beta} 
\end{bmatrix}
\begin{bmatrix}
  i_{\beta} 
\end{bmatrix}
\]

or in a more concise form,
\[ i_{\psi} = C_{\psi\beta} i_{\beta} \]

The mesh impedance matrix of the original network can be established from the branch impedance matrix of the primitive network and a connection matrix, using equation (4.7). It may be shown, by a process of induction, that the same equation applies when the primitive network is represented by compound matrices. Hence in this case,
\[ Z_{\beta\beta} = C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta} \quad (4.7) \]

Using compound matrix algebra, \( Z_{\beta\beta} \) can now be evaluated thus:
\[
Z_{\beta\beta} = C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta} = \begin{bmatrix}
  C_{\beta\delta}^t : C_{\beta\theta}^t \\
\end{bmatrix}
\begin{bmatrix}
  Z_{\delta\delta} & 0 \\
  0 & Z_{\theta\theta} 
\end{bmatrix}
\begin{bmatrix}
  C_{\delta\beta} \\
  \vdots \\
  C_{\theta\beta} 
\end{bmatrix}
\]
Linear Transformations

\[ = C_{\beta \delta}^{\beta} \cdot Z_{\delta \delta} \cdot C_{\delta \beta}^{\beta} + C_{\beta \theta}^{\beta} \cdot Z_{\theta \theta} \cdot C_{\theta \beta}^{\beta} \]

\[
C_{\beta \delta}^{\beta} \cdot Z_{\delta \delta} \cdot C_{\delta \beta}^{\beta} = \begin{bmatrix}
1 & . & \ldots & (Z_1 + Z_4) & -Z_4 & \ldots \\
. & 1 & \ldots & -Z_4 & Z_4 & \ldots \\
. & . & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

\[ = \begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & . \\
-Z_4 & Z_4 & . \\
. & . & \ldots
\end{bmatrix} \]

\[
C_{\beta \theta}^{\beta} \cdot Z_{\theta \theta} \cdot C_{\theta \beta}^{\beta} = \begin{bmatrix}
. & . & \ldots & (Z_2 + Z_5) & -Z_5 & \ldots \\
-1 & . & \ldots & -Z_5 & (Z_2 + Z_5) & . \\
. & -1 & . & \ldots & \ldots & \ldots
\end{bmatrix}
\]

\[ = \begin{bmatrix}
. & . & \ldots & \ldots & \ldots \\
. & (Z_2 + Z_5) & -Z_5 \\
. & -Z_5 & (Z_3 + Z_5)
\end{bmatrix} \]

Hence,

\[ Z_{\beta \beta} = \begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & . \\
-Z_4 & (Z_2 + Z_4 + Z_5) & -Z_5 \\
. & -Z_5 & (Z_3 + Z_5)
\end{bmatrix} \]

The final equations for the original network can now be established by constructing the two vectors \( i_\beta \) and \( E_\beta \) by inspection.

\[
\begin{bmatrix}
(Z_1 + Z_4) & -Z_4 & . \\
-Z_4 & (Z_2 + Z_4 + Z_5) & -Z_5 \\
. & -Z_5 & (Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix} = \begin{bmatrix}
E_p \\
. \\
-\bar{E}_r
\end{bmatrix}
\]

Further applications of linear transformation can be found in
the section dealing with symmetrical components. The use of linear transformation in solving functional equations is shown in chapter 5 onwards.

**Networks with transformers**

In general engineering analysis it is sometimes necessary to solve problems which can be represented by electrical networks containing transformers.

The inclusion of an ideal transformer linking two previously separate circuits introduces a linear relationship between the primary and secondary currents. If these currents are expressed in terms of mesh currents, a linear relationship between the mesh currents is obtained, which enables one mesh current to be expressed in terms of the remaining mesh currents so that the number of unknown mesh currents is reduced by one for each transformer present.

Consider, for example, the network shown in fig. 4.4(a) with three transformers x, y, z. In the particular example there is one closed magnetic mesh for each transformer as shown in fig. 4.4(b). The analysis for networks with transformers is carried out in a similar manner to that for networks without them, but with an additional equation for each closed magnetic mesh.

**Mesh current relationship**

For the particular example there are three closed magnetic meshes and, neglecting the magnetizing current by assuming that the sum of the m.m.f.'s around a closed magnetic mesh is zero, the following three relationships may be set up by inspection of fig. 4.4(b).

\[ N_7 i_7 + N_8 i_8 = 0 \]
\[ N_9 i_9 + N_{10} i_{10} = 0 \]
\[ N_{11} i_{11} + N_{12} i_{12} = 0 \]

where \( N_j (j = 7, 8, ..., 12) \) is the number of turns on winding j.

The currents in the windings may be expressed in terms of the defined mesh currents, therefore the above relationships may be written by inspection of fig. 4.4(a) as
By means of the above equations certain of the mesh currents may be expressed in terms of the remaining mesh currents. The former are called dependent currents and the latter independent currents.
The choice of independent currents is not completely arbitrary. In the example considered, \( i_q \) does not flow through any transformer winding and so does not appear in equations (4.8) hence \( i_q \) cannot be expressed in terms of the other mesh currents.

Equations (4.8) enable any three of the four currents \( i_p, i_r, i_s, i_t \) to be expressed in terms of the remaining currents. In the particular example \( i_p, i_q \) are taken as the independent currents and \( i_r, i_s, i_t \) as the dependent currents. The subscript \( \delta (\delta = p, q) \) is associated with the independent currents. All the branch currents may be expressed in terms of the two currents \( i_p, i_q \). When the currents are expressed in terms of \( i_p, i_q \) only, the notation \( i'_p, i'_q \) will be used. The reason for the distinction by means of primes will become apparent later.

\[
i_p = i'_p \\
i_q = i'_q
\]

Since for any one closed magnetic mesh the sum of m.m.f.'s is zero, it is not necessary to specify the number of turns for each winding; it is sufficient to use the relative number of turns for one winding with respect to the other, i.e. the turns ratio. From equation (4.8x)

\[
i_s = -\frac{N_7}{N_8} i_p = N_{sp} i'_p \quad \text{where} \quad N_{sp} = -\frac{N_7}{N_8}
\]

From equation (4.8z)

\[
i_r = -\frac{N_{10}}{N_{11}} i_t = N_{rt} i_t \quad \text{where} \quad N_{rt} = -\frac{N_{12}}{N_{11}}
\]

Substituting for \( i_r \) into equation (4.8y)

\[
N_9 i_p + (N_{rt} N_{10} - N_9 - N_{10}) i_t = 0
\]

Hence

\[
i_t = \frac{-N_9}{N_{rt} N_{10} - N_9 - N_{10}} i_p = N_{tp} i'_p
\]

where

\[
N_{tp} = \frac{-N_9}{N_{rt} N_{10} - N_9 - N_{10}}
\]

and

\[
i_r = N_{rt} i_t = N_{rt} N_{ip} i'_p = N_{rp} i'_p
\]

where

\[
N_{rp} = N_{rt} N_{tp}
\]
The above relationships may be written in matrix form as

$$
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
i_t
\end{bmatrix} = \begin{bmatrix} 1 & . \\ . & 1 \\ N_{rp} & . \\ N_{sp} & . \\ N_{tp} & . \end{bmatrix} \begin{bmatrix} i'_p \\ i'_q \\
i'_r \\
i'_s \\
i'_t
\end{bmatrix}
$$

or in general terms as

$$
i_\beta = C_{\beta \delta} i'_\delta
$$

(4.9)

The matrix $C_{\beta \delta}$ relates all the defined mesh currents in terms of the independent mesh currents. This matrix equation is easier to understand by having retained the separate identities of $i_p, i'_p$ and $i_q, i'_q$.

Transformation law

The functional equation describing the problem in terms of the independent mesh currents is

$$Z_{\delta \delta} i'_\delta = E'_\delta$$

where $E'_\delta$ and $Z_{\delta \delta}$ are the effective mesh voltage and mesh impedance matrices respectively. This may be established from the following consideration:

Using equations (4.2), (4.3), (4.4)

$$Z_{\psi \psi} i_\psi = e_\psi \quad (\psi = 1, 2, 3, 4, 5, 6) \quad (4.2)$$

$$i_\psi = C_{\psi \beta} i_\beta \quad (4.3)$$

and from conservation of power

$$E_{\beta \psi}^* i_\beta^* = e_\psi^* i_\psi^* \quad (4.10)$$

Substituting for $i_\beta$ from equation (4.9) into equation (4.3),

$$i_\psi = C_{\psi \beta} C_{\beta \delta} i'_\delta$$

Let

$$C_{\psi \delta} = C_{\psi \beta} C_{\beta \delta} \quad (4.11)$$
then

\[ i_\varphi = C_{\psi \delta} i_\delta \tag{4.12} \]

Substituting for \( i_\beta \) from equation (4.9) and for \( i_\psi \) from equation (4.12) into equation (4.10)

\[ E_\beta^t C_{\beta \delta}^* i_\delta^* = e_\psi^t C_{\psi \delta}^* i_\delta^* \]

Since \( i_\delta^* \) is an arbitrary vector and can take any value, it follows from chapter 2 that

\[ E_\beta^t C_{\beta \delta}^* = e_\psi^t C_{\psi \delta}^* \]

Transposing,

\[ C_{\delta \beta}^* E_\beta = C_{\delta \psi}^* e_\psi \]

Substituting for \( e_\psi \) from equation (4.2),

\[ C_{\delta \beta}^* E_\beta = C_{\delta \psi}^* Z_{\psi \delta} i_\psi \]

Substituting for \( i_\psi \) from equation (4.12),

\[ C_{\delta \beta}^* E_\beta = C_{\delta \psi}^* Z_{\psi \delta} C_{\psi \delta} i_\delta^* \]

or as

\[ E_{\delta}^t = Z_{\delta \delta}^t i_{\delta}^t \]

where

\[ E_{\delta} = C_{\delta \beta}^* E_\beta \]

and

\[ Z_{\delta \delta} = C_{\delta \psi}^* Z_{\psi \delta} C_{\psi \delta} \tag{4.14} \]

An alternative development of the transformation law may be based on topological considerations using both Kirchhoff's laws.

**Summary of steps**

1. Establish the connection matrix \( C_{\psi \beta} \) by expressing individual branch currents in terms of the defined mesh currents.
2. Establish the connection matrix \( C_{\beta \delta} \) by expressing the defined mesh currents in terms of the independent currents.
3. Evaluate the matrix \( C_{\psi \delta} \) from equation (4.11)

\[ C_{\psi \delta} = C_{\psi \beta} C_{\beta \delta} \]

4. Evaluate the effective voltage source \( E_{\delta}^t \) from the known voltage sources \( E_\beta \) using equation (4.13)
**Linear Transformations**

\[ \mathbf{E}^t_\delta = \mathbf{C}^{*t}_\delta \beta \mathbf{E}_\beta \]

(5) Evaluate the effective mesh impedance matrix \( \mathbf{Z}^t_\delta \delta \) from the branch impedance matrix \( \mathbf{Z}_\psi \psi \) using equation (4.14)

\[ \mathbf{Z}^t_\delta \delta = \mathbf{C}^{*t}_\delta \psi \mathbf{Z}_\psi \psi \mathbf{C}_\psi \delta \]

(6) The equation

\[ \mathbf{Z}^t_\delta \delta \mathbf{i}^t_\delta = \mathbf{E}^t_\delta \]

may now be solved for \( \mathbf{i}^t_\delta \) to give

\[ \mathbf{i}^t_\delta = \mathbf{Z}^{t-1}_\delta \delta \mathbf{E}^t_\delta \]

and the individual mesh currents obtained using equation (4.9)

\[ \mathbf{i}_\beta = \mathbf{C}_\beta \delta \mathbf{i}^t_\delta \]

**Example**

Consider the network shown in fig. 4.5 where the turns ratio for the two transformers is given as

\[ \frac{N_7}{N_8} = \frac{1}{\lambda} \quad \text{and} \quad \frac{N_9}{N_{10}} = \frac{1}{\lambda^2} \]

---

**Fig. 4.5**
where $\lambda$ is a complex number

$$\lambda = \cos 120^\circ + j\sin 120^\circ = -0.5 + j0.866$$

hence

$$\lambda^2 = \cos 240^\circ + j\sin 240^\circ = -0.5 - j0.866$$

Therefore

$$\frac{N_7}{N_8} = \frac{1}{\lambda} = \frac{v_7}{v_8} = \frac{i^*_s}{i^*_p} = \frac{i^*_s}{i^*_p} \quad \text{and} \quad \frac{N_9}{N_{10}} = \frac{1}{\lambda^2} = \frac{v_9}{v_{10}} = \frac{i^*_g}{i^*_t} = \frac{i^*_g}{i^*_t}$$

Problems of this type arise in solving three-phase networks with simultaneous unbalanced faults. The various matrices may be constructed by inspection of fig. 4.5 as

$$E_\beta = \begin{bmatrix} E_p \\ E_q \\ E_r \\ E_s \end{bmatrix} \quad i_\beta = \begin{bmatrix} i_p \\ i_q \\ i_r \\ i_s \end{bmatrix} \quad i'_\delta = \begin{bmatrix} i'_p \\ i'_q \\ i'_r \\ i'_s \end{bmatrix}$$


Following the summary of steps shown on p. 104, the following matrices are established.

(1) Expressing individual branch currents in terms of the defined mesh currents,

$$i_1 = i_p \quad i_4 = i_q + i_s$$

$$i_2 = i_p + i_q \quad i_5 = i_s$$

$$i_3 = i_q + i_r, \quad i_6 = i_r$$

or in matrix form as
Linear Transformations

\[
\begin{bmatrix}
  i_1 \\
  i_2 \\
  i_3 \\
  i_4 \\
  i_5 \\
  i_6 \\
\end{bmatrix} = \begin{bmatrix}
  1 & \ldots & . \\
  1 & 1 & \ldots \\
  . & 1 & 1 \\
  . & 1 & . \\
  . & . & 1 \\
  . & . & 1 \\
\end{bmatrix} \begin{bmatrix}
  i_p \\
  i_q \\
  i_r \\
  i_s \\
\end{bmatrix}
\]

hence

\[
\mathbf{C}_{\psi\beta} = \begin{bmatrix}
  1 & \ldots & . \\
  1 & 1 & \ldots \\
  . & 1 & 1 \\
  . & 1 & . \\
  . & . & 1 \\
  . & . & 1 \\
\end{bmatrix}
\]

(2) Choosing \( i_p \) and \( i_q \) as the independent currents and expressing the mesh currents in terms of the independent currents,

\[
\begin{align*}
  i_p &= i_p' \\
  i_q &= i_q' \\
  i_r &= \lambda^2 i_p' \\
  i_s &= \lambda i_p'
\end{align*}
\]
or in matrix form as

\[
\begin{bmatrix}
  i_p \\
  i_q \\
  i_r \\
  i_s \\
\end{bmatrix} = \begin{bmatrix}
  1 & . \\
  . & 1 \\
  \lambda^2 & . \\
  \lambda & . \\
\end{bmatrix} \begin{bmatrix}
  i_p' \\
  i_q' \\
  \lambda^2 \\
  \lambda \\
\end{bmatrix}
\]

hence

\[
\mathbf{C}_{\beta\beta} = \begin{bmatrix}
  1 & . \\
  . & 1 \\
  \lambda^2 & . \\
  \lambda & . \\
\end{bmatrix} \quad \text{and} \quad \mathbf{C}_{\beta\beta}^* = \begin{bmatrix}
  1 & \lambda & \lambda^2 \\
  . & 1 & . \\
\end{bmatrix}
\]

Note that \( \lambda^* = \lambda^2 \) and \( (\lambda^2)^* = \lambda \)
(3) \( \mathbf{C}_{\psi \delta} = \mathbf{C}_{\psi \beta} \mathbf{C}_{\beta \delta} \)

\[
\mathbf{C}_{\psi \beta} \mathbf{C}_{\beta \delta} = \begin{bmatrix}
1 & . & . & . \\
1 & 1 & . & . \\
. & 1 & 1 & . \\
. & 1 & . & 1 \\
. & . & 1 & . \\
\end{bmatrix}
\begin{bmatrix}
1 & . \\
1 & 1 \\
\lambda^2 & . \\
\lambda & . \\
\lambda^2 & . \\
\end{bmatrix} =
\begin{bmatrix}
1 & . \\
1 & 1 \\
\lambda^2 & 1 \\
\lambda & 1 \\
\lambda & . \\
\lambda^2 & . \\
\end{bmatrix}
\]

hence

\[
\mathbf{C}_{\psi \delta} = \begin{bmatrix}
1 & . \\
1 & 1 \\
\lambda^2 & 1 \\
\lambda & 1 \\
\lambda & . \\
\lambda^2 & . \\
\end{bmatrix}
\]

\[
\mathbf{C}_{\delta \psi}^{\ast t} = \begin{bmatrix}
1 & 1 & \lambda & \lambda^2 & \lambda^2 & \lambda \\
. & 1 & 1 & 1 & . & . \\
\end{bmatrix}
\]

(4) The effective voltage source \( \mathbf{E}_\delta^t \) is evaluated from

\[
\mathbf{E}_\delta^t = \mathbf{C}_{\delta \beta}^\ast \mathbf{E}_\beta
\]

\[
\begin{bmatrix}
1 & \lambda & \lambda^2 \\
. & 1 & . \\
\end{bmatrix}
\begin{bmatrix}
E_p \\
E_q \\
E_r \\
E_s \\
\end{bmatrix} =
\begin{bmatrix}
E_p + \lambda E_r + \lambda^2 E_s \\
E_q \\
\end{bmatrix}
\]

(5) The effective mesh impedance matrix \( \mathbf{Z}_{\delta \delta}^\ast \) is evaluated from \( \mathbf{Z}_{\delta \delta}^\prime = \mathbf{C}_{\delta \psi}^{\ast t} \mathbf{Z}_{\psi \psi} \mathbf{C}_{\psi \delta} \) in two stages. Multiplying \( \mathbf{Z}_{\psi \psi} \mathbf{C}_{\psi \delta} \).
Linear Transformations

\[
\begin{bmatrix}
Z_1 & \cdots & \cdots & \cdots \\
. & Z_2 & \cdots & \cdots \\
. & . & Z_3 & \cdots \\
. & . & . & Z_4 \\
. & . & . & . & Z_5 \\
. & . & . & . & . & Z_6 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\lambda^2 \\
\lambda \\
\lambda^2 \\
\lambda^2 \\
\end{bmatrix}
= \begin{bmatrix}
Z_1 \\
Z_2 & Z_2 \\
\lambda^2 Z_3 & Z_3 \\
\lambda Z_4 & Z_4 \\
\lambda Z_5 \\
\lambda^2 Z_6 \\
\end{bmatrix}
\]

Multiplying $C^*_\delta \psi (Z_\psi \psi C_{\psi \beta})$

\[
\begin{bmatrix}
1 & 1 & \lambda & \lambda^2 & \lambda^2 & \lambda \\
1 & 1 & 1 & 1 & \cdots \\
\lambda^2 Z_3 & Z_3 \\
\lambda Z_4 & Z_4 \\
\lambda Z_5 \\
\lambda^2 Z_6 \\
\end{bmatrix}
\begin{bmatrix}
Z_1 \\
Z_2 & Z_2 \\
\lambda^2 Z_3 & Z_3 \\
\lambda Z_4 & Z_4 \\
\lambda Z_5 \\
\lambda^2 Z_6 \\
\end{bmatrix}
= \begin{bmatrix}
(Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6) & (Z_2 + \lambda Z_3 + \lambda^2 Z_4) \\
(Z_2 + \lambda^2 Z_3 + \lambda Z_4) & (Z_2 + Z_3 + Z_4) \\
\end{bmatrix}
\]

(6) The required final equations can now be constructed as

\[
Z^I_\delta i^I_\delta = E^I_\delta
\]

or in detail as

\[
\begin{bmatrix}
(Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6) & (Z_2 + \lambda Z_3 + \lambda^2 Z_4) \\
(Z_2 + \lambda^2 Z_3 + \lambda Z_4) & (Z_2 + Z_3 + Z_4) \\
\end{bmatrix}
\begin{bmatrix}
i'_p \\
i'_q \\
\end{bmatrix}
= \begin{bmatrix}
E_p + \lambda E_q + \lambda^2 E_\delta \\
E_q \\
\end{bmatrix}
\]

Nodal voltage analysis

In chapter 2 it was shown that a branch voltage source may be replaced by a hypothetical nodal current. This does not change the nature of the problem but it may result in a simpler set of equations. The reasoning and the method developed in this
chapter for formulating the mesh current equations are equally applicable to the nodal voltage analysis.

(a) original network

(b) primitive network

Fig. 4.6

To illustrate the method, consider the network shown in fig. 4.6(a). For simplicity assume that there is no mutual coupling between any of the branch admittances. In chapter 2 on p. 39 it was shown that the equations which describe the performance of the network can be set up as

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 & . \\
-Y_1 & (Y_1 + Y_2 + Y_4) & -Y_2 \\
. & -Y_2 & (Y_2 + Y_5)
\end{bmatrix}
\begin{bmatrix}
v_a \\ v_b \\ v_c
\end{bmatrix}
= \begin{bmatrix}
I_a \\ I_b \\ I_c
\end{bmatrix}
\]
Using general terms and subscripts, the above equation can be written in concise form as

\[ Y_{aa} v_a = i_a \]  (4.15)

**Primitive network**

Consider now the network shown in fig. 4.6(b). This network is constructed by breaking up the original network into individual branch components. Each individual branch admittance is supplied with a hypothetical current source which gives rise to the same voltage across the branch as in the original network. The component network is called the **primitive network**.

In the primitive network shown in fig. 4.6(b) the current flowing through branch \( Y_j \) is \( i_j \) (\( j = 1, 2, \ldots, 5 \)), which is the same as through branch \( Y_j \) of the original network shown in fig. 4.6(a). It follows therefore that the hypothetical current source and the branch current \( i_j \) are numerically equal. Similarly the voltage across the branch in the primitive network and the voltage \( v_j \) across the corresponding branch in the original network are also numerically equal.

The equations describing the performance of the primitive network can be established very simply by Ohm's law.

\[ Y_1 v_1 = i_1 \]
\[ Y_2 v_2 = i_2 \]
\[ Y_3 v_3 = i_3 \]
\[ Y_4 v_4 = i_4 \]
\[ Y_5 v_5 = i_5 \]

The above equations may be written in matrix form as

\[
\begin{bmatrix}
Y_1 & \cdots & \cdots & \cdots \\
\vdots & Y_2 & \cdots & \cdots \\
\vdots & \vdots & Y_3 & \cdots \\
\vdots & \vdots & \vdots & Y_4 \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
\end{bmatrix}
\]

The dots indicate zero elements.
The matrix equation may be written more concisely in general terms using subscripts as

$$Y_{\psi \psi} v_{\psi} = i_{\psi} \quad (4.16)$$

The branch admittance matrix $Y_{\psi \psi}$ can be established very simply by inspection. It consists of all branch admittance elements listed on the diagonal and all other elements of the matrix are zero.

The nodal admittance matrix of the original network $Y_{\alpha \alpha}$ can be obtained from the branch admittance matrix of the primitive network $Y_{\psi \psi}$ by a routine procedure to be described in the following two sections. Once the matrix $Y_{\alpha \alpha}$ is established, the equations of the original network $Y_{\alpha \alpha} v_\alpha = i_\alpha$ can be constructed by completing the vectors $v_\alpha$ and $i_\alpha$.

**Connection matrix**

The voltages in the primitive network may be expressed in terms of the nodal voltages defined in the original network. From fig. 4.6(a)

$$v_1 = v_a - v_b$$
$$v_2 = v_b - v_c$$
$$v_3 = v_a$$
$$v_4 = v_b$$
$$v_5 = v_c$$

The above relations may be written in matrix form as

$$\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{bmatrix} = \begin{bmatrix} 1 & -1 & . \\
. & 1 & -1 \\
1 & . & . \\
. & 1 & . \\
. & . & 1
\end{bmatrix} \begin{bmatrix} v_a \\
v_b \\
v_c \\
v_a \\
v_c
\end{bmatrix}$$

or in general terms as

$$v_{\psi} = C_{\psi \alpha} v_\alpha \quad (4.17)$$

Since there are no mutual admittances in the network, the
positive direction of current flowing through the branches can be assumed arbitrarily. If the assumed positive direction of current is changed, then the polarity sense of the voltage across the branch will also change and therefore the corresponding elements of the matrix $C_{\psi a}$ will change sign.

The rectangular matrix $C_{\psi a}$ is known as the connection matrix or node incidence matrix. The elements of $C_{\psi a}$ may be defined in terms of branch directions in a complementary manner to the definition of $C_{\psi \beta}$ as shown on p. 87. The element $C_{ja}$ in the row $j$ and column $a$ of the matrix $C_{\psi a}$ corresponding to the branch $j$ and node $a$ may be defined as

$$C_{ja} = \begin{cases} +1, & \text{if the branch } j \text{ is directed away from node } a. \\ -1, & \text{if the branch } j \text{ is directed towards node } a. \\ 0, & \text{if the branch } j \text{ is not connected to node } a. \end{cases}$$

Using the definition, the elements of the connection matrix can be established by inspection. Since the elements of $C_{\psi a}$ consist of $+1$, $-1$ or $0$, then any multiplication by $C_{\psi a}$ resolves to simple summation only. Details of this are given in chapter 7.

Transformation law

The transformation law relating the admittance matrix $Y_{aa}$ of the original network and the admittance matrix $Y_{\psi \psi}$ of the primitive network may be developed using the principle of power conservation or by a topological method. In order to show the concepts involved, both methods are developed in the following two sections.

Power conservation. The transformation law may be developed using the following equations derived in the previous section.

$$Y_{aa}v_a = I_a \quad (4.15)$$

$$Y_{\psi \psi}v_\psi = i_\psi \quad (4.16)$$

$$v_\psi = C_{\psi a}v_a \quad (4.17)$$

Substituting for $v_\psi$ from equation (4.17) into equation (4.16) results in

$$Y_{\psi \psi}C_{\psi a}v_a = i_\psi \quad (4.18)$$

An additional equation is required to complete the transformation
and this may be established from the following consideration. The total instantaneous power supplied to the network from the current sources may be expressed, as shown in chapter 2, as \( v^t \alpha l^*_\alpha \).

The total instantaneous power dissipated by the network may be expressed as \( v^t \psi i^*_\psi \), where \( t \) indicates matrix transposition, and \( * \) indicates complex conjugate.

Since there are no other sources or sinks of power then, by conservation of power,

\[
v^{t}_\psi i^{*}_\psi = v^{t}_\alpha l^{*}_\alpha
\]

Substituting for \( v^t_\psi \) in the above equation from equation (4.17),

\[
v^{t}_\alpha C^{t}_\alpha i^{*}_\psi = v^{t}_\alpha l^{*}_\alpha
\]

Since \( v_\alpha \) is an arbitrary vector and can take any value, it follows from chapter 1 that

\[
C^{t}_\alpha i^{*}_\psi = l^{*}_\alpha
\]

Therefore by taking complex conjugate,

\[
C^{*t}_\alpha i^{*}_\psi = l_\alpha
\]

Substituting for \( i_\psi \) in the above equation from equation (4.18),

\[
C^{*t}_\alpha Y_{\psi \psi} C_{\psi \alpha} v_\alpha = l_\alpha
\]

Comparing with equation (4.15),

\[
Y_{\alpha \alpha} = C^{*t}_\alpha Y_{\psi \psi} C_{\psi \alpha} \quad (4.19)
\]

**Topological method.** The transformation law may also be developed using the same initial equations as used in the power conservation derivation, namely

\[
Y_{\alpha \alpha} v_\alpha = l_\alpha \quad (4.15)
\]

\[
Y_{\psi \psi} v_\psi = i_\psi \quad (4.16)
\]

\[
v_\psi = C_{\psi \alpha} v_\alpha \quad (4.17)
\]

\[
Y_{\psi \psi} C_{\psi \alpha} v_\alpha = i_\psi \quad (4.18)
\]

The additional equation which replaces the conservation of power relationship is obtained from the relationship between the current sources at the individual nodes and the currents in the branches connected to the respective nodes. From fig. 4.6(a),
Linear Transformations

\[ i_1 + i_3 = I_a \]
\[ -i_1 + i_2 + i_4 = I_b \]
\[ -i_2 + i_5 = I_c \]

The above relationships may be written in matrix form as

\[
\begin{bmatrix}
1 & 1 & 1 & . & . \\
-1 & 1 & . & 1 & . \\
. & -1 & . & . & 1 \\
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
\end{bmatrix}
= \begin{bmatrix}
I_a \\
I_b \\
I_c \\
\end{bmatrix}
\]

or in general terms as

\[ B_{\alpha \psi} i_\psi = I_\alpha \] (4.20)

The matrix \( B_{\alpha \psi} \) consists of as many rows as there are independent nodes and as many columns as there are branches in the network. The element \( B_{\alpha j} \) corresponding to the node \( \alpha \) and branch \( j \) may be defined as

\[ B_{\alpha j} = \begin{cases} 
+1, \text{ if the branch } j \text{ is directed away from node } \alpha. \\
-1, \text{ if the branch } j \text{ is directed towards node } \alpha. \\
0, \text{ if the branch } j \text{ is not connected to the node } \alpha. 
\end{cases} \]

It can be seen by comparing the definition for \( C_{\psi \alpha} \) shown on p. 113 with the above definition for \( B_{\alpha \psi} \) that the two matrices are simply related

\[ B_{\alpha \psi} = C_{\alpha \psi}^t \]

Therefore equation (4.20) may be expressed as

\[ C_{\alpha \psi}^t i_\psi = I_\alpha \]

Substituting in the above equation for \( i_\psi \) from equation (4.18) results in

\[ C_{\alpha \psi}^t Y_{\psi \psi} C_{\psi \alpha} V_\alpha = I_\alpha \]

Comparing with equation (4.15),

\[ Y_{\alpha \alpha} = C_{\alpha \psi}^t Y_{\psi \psi} C_{\psi \alpha} \] (4.21)
Although $C_{\alpha\psi}^t$ appears in equation (4.19) and $C_{\alpha\psi}^t$ appears in equation (4.21), it must be noted that, for the applications considered in this book, the elements of $C_{\psi\alpha}$ are real numbers consisting of $+1, -1, 0$, therefore

$$C_{\psi\alpha}^* = C_{\psi\alpha}$$

and

$$C_{\alpha\psi}^t = C_{\alpha\psi}^t$$

hence equations (4.19) and (4.21) take identical form.

**Example**

Consider the network shown in fig. 4.6(a) and fig. 4.6(b). The admittance matrix $Y_{\psi\psi}$ and the connection matrix $C_{\psi\alpha}$ can be constructed by inspection as

$$Y_{\psi\psi} = \begin{bmatrix}
Y_1 & . & . & . \\
. & Y_2 & . & . \\
. & . & Y_3 & . \\
. & . & . & Y_4 \\
. & . & . & Y_5
\end{bmatrix}, \quad C_{\psi\alpha} = \begin{bmatrix}
1 & 1 & -1 & . \\
2 & . & 1 & -1 \\
3 & 1 & . & . \\
4 & 1 & . & . \\
5 & . & . & 1
\end{bmatrix}$$

Using equation (4.21) and performing the multiplication $C_{\alpha\psi}^t Y_{\psi\psi}$,

$$\begin{bmatrix}
1 & . & 1 & . \\
-1 & 1 & 1 & . \\
. & -1 & . & 1
\end{bmatrix} \begin{bmatrix}
Y_1 & . & . & . \\
. & Y_2 & . & . \\
. & . & Y_3 & . \\
. & . & . & Y_4 \\
. & . & . & Y_5
\end{bmatrix}$$

$$= \begin{bmatrix}
Y_1 & . & Y_3 & . & . \\
-Y_1 & Y_2 & . & Y_4 & . \\
. & -Y_2 & . & . & Y_5
\end{bmatrix}$$

Multiplying $(C_{\alpha\psi}^t Y_{\psi\psi}) C_{\psi\alpha}$ results in the matrix $Y_{\alpha\alpha}$ which is
the nodal admittance coefficient matrix of the original network.

\[
\begin{bmatrix}
Y_1 & Y_3 & \cdots & \cdots \\
-1 & Y_2 & Y_4 & \cdots \\
\cdots & -Y_2 & \cdots & Y_5 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & \cdots \\
\cdots & 1 & -1 \\
1 & \cdots \\
1 & \cdots \\
\end{bmatrix}
= \begin{bmatrix}
(Y_1 + Y_3) & -Y_1 \\
-Y_1 & (Y_1 + Y_2 + Y_4) & -Y_2 \\
\cdots & -Y_2 & (Y_2 + Y_5) \\
\end{bmatrix}
\]

In the above simple example the ordinary method of forming equations is faster since the matrix \(Y_{aa}\) can be written down directly and easily by inspection. As the complexity of the network increases, the linear transformation method becomes more convenient.

**Interconnection of networks**

In order to write the equations for large and complex networks it is not necessary to subdivide into individual branches. It is sufficient to subdivide into a number of smaller networks, to write down the equations for each individual network independently and then to transform them into the equations of the original network.

Consider the network shown in fig. 4.7(a) subdivided arbitrarily into two parts. The primitive network is obtained by injecting hypothetical nodal currents at each node as shown in fig. 4.7(b). The equations of the individual networks may be written by inspection as

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 \\
-1 & Y_1 \\
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
\end{bmatrix}
= \begin{bmatrix}
v_1 \\
v_2 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
(Y_2 + Y_4) & -Y_2 \\
-1 & (Y_2 + Y_5) \\
\end{bmatrix}
\begin{bmatrix}
i_3 \\
i_4 \\
\end{bmatrix}
= \begin{bmatrix}
v_3 \\
v_4 \\
\end{bmatrix}
\]
or in general terms as

\[ Y_{ss} i_s = v_s \quad \text{and} \quad Y_{ss} i_s = v_s \]

(a) original network

(b) primitive network

Fig. 4.7

Combining the two sets of equations into one compound matrix equation,

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 & & & \\
-Y_1 & Y_1 & & & \\
& & \ddots & & \\
0 & & & (Y_2 + Y_4) & -Y_2 \\
& & & -Y_2 & (Y_2 + Y_5)
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4
\end{bmatrix}
= 
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
\]
which may be written in general terms as
\[
\begin{bmatrix}
Y_{\delta\delta} & : & 0 \\
\vdots & & \vdots \\
0 & : & Y_{\alpha\alpha}
\end{bmatrix}
\begin{bmatrix}
i_\delta \\
\vdots \\
i_\alpha
\end{bmatrix}
= 
\begin{bmatrix}
v_\delta \\
\vdots \\
v_\alpha
\end{bmatrix}
\]

or in a more concise form as
\[
Y_{\psi\psi} i_\psi = v_\psi
\]

The voltages across the individual branches are related to the defined nodal voltages from fig. 4.7(a) and fig. 4.7(b) as
\[
\begin{align*}
v_1 &= v_a \\
v_2 &= v_b \\
v_3 &= v_b \\
v_4 &= v_c
\end{align*}
\]

and in matrix form as
\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
= 
\begin{bmatrix}
1 & : & . \\
. & & 1 \\
. & & 1 \\
. & & .
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\]

or in compound matrix form
\[
\begin{bmatrix}
v_\delta \\
\vdots \\
v_\alpha
\end{bmatrix}
= 
\begin{bmatrix}
C_{\delta\alpha} \\
\vdots \\
C_{\alpha\alpha}
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\]

or in a more concise form
\[
v_\psi = C_{\psi\alpha} v_\alpha
\]

The admittance matrix $Y_{a\alpha}$ of the original network can be established from the admittance matrix $Y_{\psi\psi}$ of the primitive network and the connection matrix $C_{\psi\alpha}$ using equation (4.21)
\[
Y_{a\alpha} = C_{a\psi}^i Y_{\psi\psi} C_{\psi\alpha}
\]  \hspace{1cm} (4.21)

Using compound matrix algebra
$$Y_{aa} = C_{\alpha\delta}^t Y_{\psi\psi} C_{\psi\alpha} = \begin{bmatrix} C_{\delta\delta}^t & C_{\alpha\delta}^t \\ \vdots & \vdots \\ 0 & Y_{\theta\theta} \end{bmatrix} \begin{bmatrix} Y_{\delta\delta} \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ C_{\delta\psi} \\ C_{\theta\alpha} \end{bmatrix}$$

$$= C_{\alpha\delta}^t Y_{\delta\delta} C_{\delta\alpha} + C_{\alpha\theta}^t Y_{\theta\theta} C_{\theta\alpha}$$

$$C_{\alpha\delta}^t Y_{\delta\delta} C_{\delta\alpha} = \begin{bmatrix} 1 & . \\ . & 1 \\ . & . \end{bmatrix} \begin{bmatrix} (Y_1 + Y_3) & -Y_1 \\ -Y_1 & Y_1 \end{bmatrix} \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}$$

$$C_{\alpha\theta}^t Y_{\theta\theta} C_{\theta\alpha} = \begin{bmatrix} . & . \\ 1 & . \\ . & 1 \end{bmatrix} \begin{bmatrix} (Y_2 + Y_4) & -Y_2 \\ -Y_2 & (Y_2 + Y_6) \end{bmatrix} \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}$$

and

$$Y_{aa} = C_{\alpha\delta}^t Y_{\delta\delta} C_{\delta\alpha} + C_{\alpha\theta}^t Y_{\theta\theta} C_{\theta\alpha}$$

$$Y_{aa} = \begin{bmatrix} (Y_1 + Y_3) & -Y_1 \\ -Y_1 & Y_1 \end{bmatrix} + \begin{bmatrix} . & . \\ . & 1 \end{bmatrix} \begin{bmatrix} (Y_2 + Y_4) & -Y_2 \\ -Y_2 & (Y_2 + Y_6) \end{bmatrix}$$

$$= \begin{bmatrix} (Y_1 + Y_3) & -Y_1 \\ -Y_1 & (Y_1 + Y_2 + Y_4) \end{bmatrix} + \begin{bmatrix} . & . \\ . & 1 \end{bmatrix} \begin{bmatrix} (Y_2 + Y_4) & -Y_2 \\ -Y_2 & (Y_2 + Y_6) \end{bmatrix}$$

The final equations for the original network can now be
established as

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 & . \\
-Y_1 & (Y_1 + Y_2 + Y_4) & -Y_2 \\
. & -Y_2 & (Y_2 + Y_5)
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
= \begin{bmatrix}
I_a \\
I_b \\
I_c
\end{bmatrix}
\]

Duals

Comparison of the mesh current analysis and the nodal voltage analysis indicates obvious similarities. It is observed for example that equations (4.7) and (4.21) are complementary to each other. By interchanging the symbols, these two equations and all intermediate equations would be identical. The quantities and the networks described by these equations are called duals. The advantage of recognizing this dual property is that by simple reference to a table of dual quantities, the equations of nodal voltage analysis can be written down by simple analogy from the equations of mesh current analysis without duplicating the derivation of the equations. The same table can also be used to establish the equations of nodal voltage analysis from the equations of mesh current analysis.

Some dual relationships which apply in electrical networks are given in table 4.1.

\textbf{Table 4.1}

\textit{Dual electrical quantities}

<table>
<thead>
<tr>
<th>Mesh current analysis</th>
<th>Nodal voltage analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>E Voltage source</td>
<td>I Current source</td>
</tr>
<tr>
<td>i Mesh current</td>
<td>v Nodal voltage</td>
</tr>
<tr>
<td>L Inductance</td>
<td>C Capacitance</td>
</tr>
<tr>
<td>R Resistance</td>
<td>G Conductance ((G = 1/R))</td>
</tr>
<tr>
<td>C Capacitance</td>
<td>L Inductance</td>
</tr>
<tr>
<td>Z Impedance</td>
<td>Y Admittance</td>
</tr>
</tbody>
</table>

Symmetrical components

\textit{Introduction}

In the preceding part of this chapter dealing with mesh analysis a relationship of the form

\[ Z\psi\bar{\psi}i_\psi = e_\psi \]  \hspace{1cm} (4.22)
where \( \mathbf{e}_\psi \) and \( \mathbf{i}_\psi \) denote primitive voltages and currents, was transformed by expressing the primitive currents \( \mathbf{i}_\psi \) in terms of a new set of currents \( \mathbf{i}_\beta \) and introducing a new set of voltages \( \mathbf{E}_\beta \) to replace the primitive voltages \( \mathbf{e}_\psi \). The essential feature of the transformation is that each primitive current can be expressed as a linear combination of the new currents by an equation of the form

\[
\mathbf{i}_\psi = \mathbf{C}_{\psi\beta} \mathbf{i}_\beta \tag{4.23}
\]

At the same time each new voltage can be expressed as a linear combination of the primitive voltages by an equation of the form

\[
\mathbf{B}_{\beta\psi} \mathbf{e}_\psi = \mathbf{E}_\beta \tag{4.24}
\]

Substituting for \( \mathbf{e}_\psi \) in equation (4.24) from equation (4.22) and then substituting for \( \mathbf{i}_\psi \) using equation (4.23) gives an equation relating the new voltages and the new currents in terms of mesh currents

\[
\mathbf{B}_{\beta\psi} \mathbf{Z}_{\psi\psi} \mathbf{C}_{\psi\beta} \mathbf{i}_\beta = \mathbf{E}_\beta
\]

This may be written as

\[
\mathbf{Z}_{\beta\beta} \mathbf{i}_\beta = \mathbf{E}_\beta
\]

where

\[
\mathbf{Z}_{\beta\beta} = \mathbf{B}_{\beta\psi} \mathbf{Z}_{\psi\psi} \mathbf{C}_{\psi\beta}
\]

Equations (4.23) and (4.24) are simple applications of Kirchhoff’s laws and they implicitly introduce constraints which result in reduction of the number of equations describing the network.

From topological considerations \( \mathbf{B}_{\beta\psi} = \mathbf{C}_{\psi\beta}^T \) and, as a consequence, power invariance is preserved in this case.

**Symmetrical components as a linear transformation**

There are occasions when the network equations are transformed by linear transformations of the forms (4.23) and (4.24) in which the new variables are not designated as mesh (or nodal) quantities. The new currents and voltages may be fictitious and may be introduced not to reduce the number of variables but to transform the set of equations (4.22) into a more convenient form. If the transformations (4.23) and (4.24) do not introduce any further constraints, either directly or by implication the number of variables cannot be reduced and the matrices \( \mathbf{B} \) and \( \mathbf{C} \) will be non-singular. Such a situation arises from the use of symmetrical components.
in the analysis of three-phase systems.

In an unbalanced three-phase system there are three asymmetrical voltages \( E_a, E_b, E_c \). Fortesque [14] showed that these voltages can be expressed in terms of symmetrical component voltages, \( E_0, E_1 \) and \( E_2 \) as

\[
\begin{align*}
E_a &= E_0 + E_1 + E_2 \\
E_b &= E_0 + \lambda E_1 + \lambda^2 E_2 \\
E_c &= E_0 + \lambda E_1 + \lambda^2 E_2
\end{align*}
\]

where \( \lambda = \cos 120^\circ + j \sin 120^\circ = -0.5 + j0.866 \)

\( \lambda^2 = \cos 240^\circ + j \sin 240^\circ = -0.5 - j0.866 \)

The above equation may be written in matrix form as

\[
\begin{bmatrix}
E_a \\
E_b \\
E_c
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & \lambda & \lambda^2 \\
1 & \lambda & \lambda^2
\end{bmatrix}
\begin{bmatrix}
E_0 \\
E_1 \\
E_2
\end{bmatrix}
\]

or in general terms as

\[
E_\phi = C_{\phi \sigma} E_\sigma
\]

Now \( C_{\phi \sigma} \) is non-singular. In fact it is easily verified that

\[
(C_{\phi \sigma})^{-1} = \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & \lambda & \lambda^2 \\
1 & \lambda^2 & \lambda
\end{bmatrix}
= \frac{1}{3} C^{*t}_{\sigma \phi}
\]

(4.25)

and

\[
C^{*t}_{\sigma \phi} C_{\phi \sigma} = 3
\]

(4.26)

Thus any given set of unbalanced voltages \( E_\phi (\phi = a, b, c) \) can be expressed in terms of symmetrical components \( E_\sigma (\sigma = 0,1,2) \) where

\[
E_\sigma = (C_{\phi \sigma})^{-1} E_\phi = \frac{1}{3} C^{*t}_{\sigma \phi} E_\phi
\]

i.e.

\[
E_0 = \frac{1}{3} (E_a + E_b + E_c)
\]

\[
E_1 = \frac{1}{3} (E_a + \lambda E_b + \lambda^2 E_c)
\]

\[
E_2 = \frac{1}{3} (E_a + \lambda^2 E_b + \lambda E_c)
\]
In an identical way a set of three unbalanced currents \( i_\phi \) may be expressed in terms of a new symmetrical component current system \( i_\sigma \). The corresponding equations will be

\[
i_\phi = C_{\phi\sigma} i_\sigma
\]

\[
i_\sigma = (C_{\phi\sigma})^{-1} i_\phi = \frac{1}{3} C_{\sigma\phi}^{*t} i_\phi
\]  \hspace{1cm} (4.27)

**Law of transformation and power relationship**

Corresponding to the equation

\[
Z_{\phi\phi} i_\phi = E_\phi
\]

which relates the phase voltages and currents, there is a relationship

\[
Z_{\sigma\sigma} i_\sigma = E_\sigma
\]

connecting the corresponding new symmetrical component voltages and currents \( E_\sigma \) and \( i_\sigma \). To find \( Z_{\sigma\sigma} \), \( E_\phi \) and \( i_\phi \) may be eliminated using equations (4.25) and (4.27) to give

\[
(C_{\phi\sigma})^{-1} Z_{\phi\phi} C_{\phi\sigma} i_\sigma = E_\sigma
\]

or

\[
\frac{1}{3} C_{\sigma\phi}^{*t} Z_{\phi\phi} C_{\phi\sigma} i_\sigma = E_\sigma
\]

hence

\[
Z_{\sigma\sigma} = \frac{1}{3} C_{\sigma\phi}^{*t} Z_{\phi\phi} C_{\phi\sigma}
\]

The power consumed in the component system is given by

\[
S = E_\phi^t i_\phi^* = (C_{\phi\sigma} E_\sigma)(C_{\phi\sigma} i_\sigma)^* = E_\sigma^t C_{\sigma\phi} C_{\phi\sigma}^* i_\sigma^* = 3 E_\sigma^t i_\sigma^*
\]

since \( C_{\sigma\phi}^t C_{\phi\sigma} = 3 \), by taking the conjugate of equation (4.26). It will be seen that with the transformation above, which is the one in general use, the principle of power invariance does not apply. Calculations may be carried out using either the phase quantities or symmetrical components but not both simultaneously. The use of the correct inverse transform ensures that the principle of conservation of energy is preserved.

It is possible to obtain a system in which the invariance of power is valid by multiplying the quantities \( E_\sigma \) and \( i_\sigma \) by a factor \( 1/\sqrt{3} \) so that

\[
E_\sigma = \frac{1}{\sqrt{3}} C_{\sigma\phi} E_\phi
\]

\[
i_\sigma = \frac{1}{\sqrt{3}} C_{\sigma\phi} i_\phi
\]
By this means the simultaneous use of symmetrical component quantities and phase quantities is possible.

Problems

1. Find the connection matrix $C_{\psi\beta}$ and using the relation
   
   $Z_{\beta\beta} = C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta}$ or $Z_{\beta\beta} = C_{\beta\psi}^t Z_{\psi\psi} C_{\psi\beta}$ as appropriate,
   establish the equations $Z_{\beta\beta} i_{\beta} = E_{\beta}$ for the networks shown in

   (a) fig. 4.8  
   (b) fig. 4.9  
   (c) fig. 4.10  
   (d) fig. 4.11

   where $Z_{\beta\beta} = \text{coefficient matrix}$
   
   $i_{\beta} = \text{mesh currents}$
   
   $E_{\beta} = \text{voltage sources}$
   
   $Z_{\psi\psi} = \text{impedance matrix of individual branch elements}$

   Verify the results using Ohm’s and Kirchhoff’s laws.
2. Find the connection matrix $C_{\psi \alpha}$ and using the relation $Y_{aa} = C_{aa}^t Y_{\psi \psi} C_{\psi \alpha}$ establish the equations $Y_{aa} v_\alpha = l_\alpha$ for the networks shown in

(a) fig. 4.12  (b) fig. 4.13

where $Y_{aa}$ = coefficient matrix

$l_\alpha$ = nodal currents

$v_\alpha$ = nodal voltages

$Y_{\psi \psi}$ = admittance matrix of individual branch elements.

Verify the results using Ohm's and Kirchoff's laws.
CHAPTER FIVE

Basic Equations of Diakoptics

Introduction

The recent advent of high-speed digital computers with large and fast-access storage has made it practicable to solve, by numerical methods, a wide range of physical problems. Although much of the work in the field of electrical networks can be accomplished by direct or iterative solutions of the type described in chapter 3, there are a number of problems for which the iterative method is difficult to use, while at the same time the computer may be too small, too slow or just simply too expensive to use for a direct solution.

In the hybrid method, also described in chapter 3, advantage has been taken of the physical nature of the network by dividing it into a number of subnetworks where the unknowns were the nodal voltages. The removed interconnections were represented by equivalent current sources connected to the nodes at each end of these interconnections. In a similar manner a scheme could be developed for networks where the unknowns are the mesh currents and the interconnections are represented by voltage sources. In a more extended method not only individual branches but whole interconnected networks could be represented by equivalent current or voltage sources.

In the method of diakoptics, networks are subdivided in the same manner, but the solution is obtained directly by a number of definite steps without any approximations or iterations.

It is widely known that Gabriel Kron [12] is the author of a method of analysis which involves dividing (tearing) the original network into a number of component networks so that each subnetwork is isolated from the rest of the system. The matrix of coefficients for each small network is inverted independently, as if the other component networks were non-existent. The solution of the full network is then obtained from the inverted matrices of the component networks by a simple routine procedure.

The solution so obtained is an exact solution within the limitations imposed by the accuracy of the parameters of the original problem and the number of digits used in the calculation.
No approximations or iterations are carried out at any stage. The method is general, applicable to a wide variety of physical problems, both electrical and mechanical, and in most cases results in considerable reduction of computational effort.

The saving arises from the fact that most physical problems are not completely interdependent, that is, any one element is not directly linked to all other elements. For example, in an electrical network there may perhaps be only two or three branches connected to any one node. The matrix equations describing such problems contain a large proportion of zero elements, and the method to be described in this chapter takes advantage of this physical property of the problem in a very convenient way.

Roth [15], Rikio Onodera [17] and Shun ichi Amari [18], by investigating diakoptics with the help of topological theory, have been able to derive the same equations as Kron without invoking laws, based on invariance of power, of the type shown in chapter 4.

For a full understanding of the development of the theory by Kron, a knowledge of tensor analysis is required, and the understanding of the development by Roth, Onodera and Amari requires some knowledge of topological concepts. As neither of these topics are generally studied by engineering students, the fundamental equations will be derived here in a simple manner using only familiar concepts of circuit topology and elementary matrix algebra. In this chapter the removal of branches not containing sources is considered, but the method is generalized to cover removal of complete sections of a network in chapter 8.

**Notation**

The notation in this chapter is based on that of matrix algebra. In order to differentiate between various matrices, simple subscript notation is adopted, as first explained in chapter 1. The reader is advised to familiarize himself with this notation, which is summarized below, before proceeding with a detailed study of this chapter.

In subscript notation, each vector or matrix will be represented by one base symbol, for example \( \mathbf{I} \), \( \mathbf{i} \), \( \mathbf{Z} \). Often the same base symbol will be used to represent different matrices and in order to distinguish one matrix from another, subscripts are used. The combination of the base letter and subscripts uniquely indentifies
a given matrix.

A vector has one subscript, e.g. $V_a$, while a matrix has two subscripts, e.g. $C_{a\beta}$. In a matrix, the subscript $a$ is related to the number of rows and the subscript $\beta$ is related to the number of columns. In a vector the subscript $a$ relates to the number of elements. The same subscript may be attached to different base letters, provided that its use is restricted to vectors having the same number of elements and to matrices having the same number of rows or columns. Thus the number of elements in the column vector $E_\beta$ and the number of columns in the matrix $C_{a\beta}$ must be the same, so that the two would conform for multiplication for the product $C_{a\beta}E_\beta$. Similarly the column vector $I_a$ transposed (shown as $I_a^t$) and the matrix $C_{a\beta}$ conform for the product $I_a^tC_{a\beta}$. The transpose of $C_{a\beta}$ will be appropriately $C_{\beta a}^t$ and the products $C_{a\beta}C_{\beta a}^t$ or $C_{\beta a}C_{a\beta}$ can be formed. It follows from this that in a matrix product the neighbouring subscripts must be identical.

The individual symbols including the subscripts are defined as they occur, but for reference purposes the basic symbols are summarized below.

\begin{align*}
C &= \text{connection matrix} \\
E &= \text{voltage sources} \\
\tilde{e} &= \text{equivalent voltage sources due to sub-division} \\
I &= \text{current sources (nodal currents)} \\
\tilde{i} &= \text{equivalent current sources due to sub-division} \\
i &= \text{mesh currents} \\
v &= \text{voltage differences (nodal voltages)} \\
Y &= \text{admittances} \\
Z &= \text{impedances}
\end{align*}

The lower case symbols $\tilde{e}$, $\tilde{i}$, $i$, $v$ signify unknown quantities and the symbols $C$, $E$, $I$, $Y$, $Z$ indicate known quantities.

The accented symbols $\tilde{i}$, and $\tilde{e}$ identify equivalent current and voltage sources due to sub-division. Similarly, $\tilde{Z}$ is the impedance matrix and $\tilde{Y}$ the admittance matrix of the equivalent networks after branches are removed. The subscripts $a, b, c, ...$
identify individual nodes; \( p, q, r, \ldots \) individual meshes; and
\( j, k, l, \ldots \) individual removed branches. The index \( \alpha \) is associ-
ated with nodal quantities, \( \beta \) with mesh quantities, and \( \psi \) with
removed branch quantities. Some of the individual items in the
equivalent subdivided network are, of course, the same as in
the original network.

Mesh current analysis

Functional equations

Consider the network shown in fig. 5.1(a) where the voltage
sources \( E_\beta \) consisting of \( E_p, E_q, E_r, \ldots, E_u \) are known and the
mesh currents \( i_\beta \) consisting of \( i_p, i_q, i_r, \ldots, i_u \) are the unknowns.
In addition, consider also the voltages \( v_\psi \) across the proposed
removed branches shown by the broken lines, consisting of \( v_j \)
and \( v_k \), as further unknowns. For the present it is assumed that
there are no current or voltage sources in the removed branches.
The problem is to obtain a set of equations from which, by elimi-
nation of the additional unknowns \( v_\psi \), a solution for the un-
known mesh currents \( i_\beta \) can be obtained.

By removing the two branches shown by broken lines in fig. 5.1
(a) and separating the remaining network at the nodes where the
two branches were connected, the original network is sub-divided
into two independent parts as shown in fig. 5.1(b). All the parts,
together with the equivalent voltage sources \( \tilde{e}_\beta \) as defined be-
low, are referred to as the equivalent network and shown in
fig. 5.1(b). All the removed branches, together with equivalent
sources \( \tilde{v}_\psi \), also defined below, are collectively referred to as
the removed network and shown in fig. 5.1(c).

The two removed branches can be represented by equivalent
hypothetical voltage sources \( \tilde{e}_\beta \) consisting of \( \tilde{e}_p, \tilde{e}_q, \tilde{e}_r, \tilde{e}_u \), of
such magnitude and sense that all the mesh currents in the
equivalent network are the same as the corresponding mesh
currents in the original network. In a complementary manner the
two component networks can be represented by equivalent hypo-
theical current sources \( \tilde{i}_\psi \) consisting of \( \tilde{i}_j, \tilde{i}_k \), of such magni-
tude and sense that all the voltages across the removed branches
are the same as the voltages across the corresponding branches
in the original network.

By inspection of fig. 5.1 it can be seen that the magnitude of
the hypothetical current sources \( \tilde{i}_\psi \) flowing into the nodes of the
\[ \tilde{Z}_{11} i_1 = E_1 + \tilde{e}_1 \]
\[ \tilde{Z}_{22} i_2 = E_2 + \tilde{e}_2 \]
\[ Z_{\beta\beta} \tilde{i}_\beta = E_\beta + \tilde{e}_\beta \]

**Fig. 5.1**
removed network must be the same as the currents flowing in the corresponding branches in the original network.

Expressing the removed branch currents in terms of the defined mesh currents for the particular example as shown in fig. 5.1 results in

\[
i_j = \tilde{i}_j = i_p - i_u \\
i_k = \tilde{i}_k = i_s - i_r
\]

These equations can be written in matrix form as

\[
\begin{bmatrix}
\tilde{i}_j \\
\tilde{i}_k
\end{bmatrix} =
\begin{bmatrix}
1 & . & . & . & . & -1 \\
. & . & -1 & 1 & . & .
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
i_t \\
i_u
\end{bmatrix}
\]

where a dot indicates zero.

The above expression can be written more concisely in general form as

\[
\tilde{i}_\psi = C_{\psi\beta} i_\beta 
\]  

(5.1)

In order to define the elements of the connection matrix \( C_{\psi\beta} \) it is convenient to make use of the branch direction concept shown in chapter 4. For any one removed branch, say branch \( j \), the corresponding row of connection matrix \( C_{\psi\beta} \) may be constructed by inspection of fig. 5.1(a). The element \( C_{jp} \) in the row \( j \) and column \( p \) of the connection matrix \( C_{\psi\beta} \) may be defined as

\[
C_{jp} = \begin{cases} 
+1, & \text{if the direction of the mesh } p \text{ coincides with the direction of the branch } j. \\
-1, & \text{if the direction of the mesh } p \text{ is opposite to the direction of the branch } j. \\
0, & \text{if the mesh } p \text{ does not include the branch } j.
\end{cases}
\]

Similarly, it can be seen from fig. 5.1, that the magnitudes of the hypothetical voltage sources \( \tilde{e}_\beta \) are the same as the voltages
Basic Equations of Diakoptics

across the removed branches which they represent with a plus or minus sign, as the case may be.

Expressing the individual hypothetical voltage sources in terms of the defined voltages across the removed branches results in

\[
\begin{align*}
\tilde{e}_p &= -v_j \\
\tilde{e}_r &= v_k \\
\tilde{e}_s &= -v_k \\
\tilde{e}_u &= v_j
\end{align*}
\]

In the particular example of fig. 5.1, branches in the path of the mesh currents \( i_q \) and \( i_t \) have not been removed. Therefore there is no need to add voltage sources in those paths. In other words, the voltage sources representing the removed branches in the \( q \) and \( t \) meshes are numerically equal to zero.

The relationship between voltage sources in every mesh, representing the removed branches and the assumed voltages across all the removed branches, may be expressed in matrix form as

\[
\begin{bmatrix}
\tilde{e}_p \\
\tilde{e}_q \\
\tilde{e}_r \\
\tilde{e}_s \\
\tilde{e}_t \\
\tilde{e}_u
\end{bmatrix} =
\begin{bmatrix}
-1 & . & . \\
. & . & . \\
. & 1 & . \\
. & . & -1 \\
. & . & . \\
1 & . & .
\end{bmatrix}
\begin{bmatrix}
v_j \\
v_k \\
v_k \\
v_k \\
v_k \\
v_k
\end{bmatrix}
\]

or in general terms as

\[
\tilde{e}_\beta = B_{\beta\gamma} v_\gamma
\]

(5.2)

In a complementary manner to the definition of the matrix \( C_{\gamma\beta} \), for the removed branch \( j \), the corresponding column of the matrix \( B_{\beta\gamma} \) may also be constructed by inspection of fig. 5.1(a). The element \( B_{pj} \) in the row \( p \) and column \( j \) of the connection matrix \( B_{\beta\gamma} \) may be defined as
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\[ B_{pj} = \begin{cases} -1, & \text{if the direction of the branch } j \text{ coincides} \\ & \text{with the direction of the mesh } p. \\ +1, & \text{if the direction of the branch } j \text{ is opposite} \\ & \text{to the direction of the mesh } p. \\ 0, & \text{if the mesh } p \text{ does not include the branch } j. \end{cases} \]

It can be seen from the above definitions that the connection matrix \( C_{\psi\beta} \) of equation (5.1) is simply related to the connection matrix \( B_{\beta\psi} \) of equation (5.2) as

\[ B_{\beta\psi} = -C_{\beta\psi}^t \]

Therefore equation (5.2) may be expressed as

\[ \tilde{e}_\beta = -C_{\beta\psi}^t \tilde{v}_\psi \tag{5.3} \]

The mesh current equations which describe the equivalent network shown in fig. 5.1(b) written in matrix form are

\[
\begin{bmatrix}
(Z_1 + Z_2 + Z_3) & -Z_2 & -Z_3 \\
-Z_2 & (Z_2 + Z_4 + Z_5) & -Z_5 \\
-Z_3 & -Z_5 & (Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix}
=
\begin{bmatrix}
E_p + \tilde{e}_p \\
E_q + 0 \\
E_r + \tilde{e}_r
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
(Z_6 + Z_7 + Z_8) & -Z_7 & -Z_5 \\
-Z_7 & (Z_7 + Z_9 + Z_{10}) & -Z_{10} \\
-Z_8 & -Z_{10} & (Z_8 + Z_{10})
\end{bmatrix}
\begin{bmatrix}
i_s \\
i_t \\
i_u
\end{bmatrix}
=
\begin{bmatrix}
E_s + \tilde{e}_s \\
E_t + 0 \\
E_u + \tilde{e}_u
\end{bmatrix}
\]

The above equations may be written in compound matrix form as

\[
\begin{bmatrix}
\tilde{Z}_{11} & \cdots & \ldots \\
\cdots & \tilde{Z}_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
=
\begin{bmatrix}
\tilde{E}_1 + \tilde{e}_1 \\
\cdots \\
\tilde{E}_2 + \tilde{e}_2
\end{bmatrix}
\]

The dots in the above matrices mean that the elements have zero values.

Writing the above equations in a general form gives:

\[ \tilde{Z}_{\beta\beta} \tilde{i}_\beta = \tilde{E}_\beta + \tilde{e}_\beta \tag{5.4} \]
The impedance matrix $\tilde{Z}_{\beta\beta}$ of the equivalent network is said to be in a block diagonal form. For computation purposes it is not necessary to store the whole matrix but only the diagonal block corresponding to the individual component networks. If two or more coefficient impedance matrices of the component networks are identical, it is necessary to store only one of them. The corresponding matrix of the original network is of the same order but is not in a block diagonal form and contains additional non-zero terms.

The equations relating the currents and voltages of the removed network as shown in fig. 5.1(c) can be written in matrix form as

$$
\begin{bmatrix}
Y_j & . \\
. & Y_k
\end{bmatrix}
\begin{bmatrix}
v_j \\
v_k
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{i}_j \\
\tilde{i}_k
\end{bmatrix}
$$

or in general form as

$$Y_{\psi\psi}v_{\psi} = \tilde{i}_{\psi}$$

Substituting for $\tilde{i}_{\psi}$ in the above equation from equation (5.1) and for $\tilde{e}_{\beta}$ in equation (5.4) from equation (5.3) results in

$$Y_{\psi\psi}v_{\psi} = C_{\psi\beta}i_{\beta}$$

(5.5)

$$\tilde{Z}_{\beta\beta}i_{\beta} = E_{\beta} - C_{\beta\psi}^t v_{\psi}$$

(5.6)

The above equations can be stated in compound matrix form as

$$
\begin{bmatrix}
Y_{\psi\psi} & -C_{\psi\beta} \\
C_{\beta\psi}^t & \tilde{Z}_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
v_{\psi} \\
i_{\beta}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
E_{\beta}
\end{bmatrix}
$$

The solution for the unknown mesh currents $i_{\beta}$ can be obtained from the above equation by eliminating the additional unknown voltages $v_{\psi}$ and this is shown in the following section. Equations (5.5) and (5.6) are the fundamental equations of diakoptics.

Equations of solution

The equations of solution for the unknown mesh currents $i_{\beta}$ may be obtained from the fundamental equations of diakoptics (5.5) and (5.6), by eliminating the additional unknown voltages $v_{\psi}$ in the following manner.
From equation (5.6),
\[ i_\beta = \tilde{Z}_{\beta\beta}^{-1}(E_\beta - C_{\beta\psi}^t v_\psi) \] (5.7)

Since \( \tilde{Z}_{\beta\beta} \) is a block diagonal matrix, \( \tilde{Z}_{\beta\beta}^{-1} \) is also a block diagonal matrix obtained by inverting the individual diagonal blocks of \( \tilde{Z}_{\beta\beta} \).

Substituting for \( i_\beta \) into equation (5.5) results in
\[ (Y_{\psi\psi} + C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} C_{\beta\psi}^t) v_\psi - C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} E_\beta = 0 \]

Let \( \tilde{Y}_{\psi\psi} = Y_{\psi\psi} + C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} C_{\beta\psi}^t \)
then
\[ v_\psi = \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} E_\beta \]

Substituting for \( v_\psi \) from the above equation into equation (5.7), the final equation of solution is obtained as

\[ i_\beta = \tilde{Z}_{\beta\beta}^{-1} E_\beta - \tilde{Z}_{\beta\beta}^{-1} C_{\beta\psi}^t \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} E_\beta \] (5.8)

A more obvious method of solving equations (5.5) and (5.6) is by direct elimination of \( v_\psi \) to give
\[ i_\beta = (\tilde{Z}_{\beta\beta} + C_{\beta\psi} Y_{\psi\psi}^{-1} C_{\psi\beta})^{-1} E_\beta \]

The above equation requires the evaluation of the inverse of \( Z_{\beta\beta} \), the full true matrix of the original network. Equations (5.8), by taking advantage of the block diagonal form of the coefficient matrix \( \tilde{Z}_{\beta\beta} \), is more economical in storage space and computation time. The true inverse matrix, if required, can always be obtained from equation (5.8) by replacing \( E_\beta \) by a unit matrix in a similar way to that considered in chapter 1.

Nodal voltage analysis

Functional equations

The fundamental equations of diakoptics, in the case where the nodal voltages are the unknowns, can be deduced by the principle of duality from the equations based on mesh current analysis. However, because of the important role that the method of nodal analysis plays in the solutions of electrical networks and to avoid any possible confusion between the two methods of analysis, the whole process of derivation is repeated.

Consider the network shown in fig. 5.2(a) where the current sources \( I_\alpha \) consisting of \( I_a, I_b, ..., I_e \) are known and the nodal
(a) original network \( Y_{aa} \mathbf{v}_a = \mathbf{i}_a \)

\[
\begin{align*}
Y_{1} & \quad I_a \\
Y_{2} & \quad V_a \\
Y_{3} & \quad \vdots \\
Y_{8} & \quad \vdots \\
Z_{i} & \quad \vdots \\
I_e & \quad \vdots \\
V_e & \quad \vdots \\
I_b & \quad \vdots \\
V_b & \quad \vdots \\
I_c & \quad \vdots \\
V_c & \quad \vdots \\
I_d & \quad \vdots \\
V_d & \quad \vdots \\
I_e & \quad \vdots \\
V_e & \quad \vdots \\
I_i & \quad \vdots \\
V_i & \quad \vdots \\
I_{\bar{i}} & \quad \vdots \\
V_{\bar{i}} & \quad \vdots \\
Z_{\bar{i}} & \quad \vdots \\
I_{\bar{e}} & \quad \vdots \\
V_{\bar{e}} & \quad \vdots \\ 
\end{align*}
\]

(b) equivalent network

\[
\begin{align*}
\tilde{Y}_{11} \mathbf{v}_1 = I_1 + \tilde{i}_1 \\
\tilde{Y}_{22} \mathbf{v}_2 = I_2 + \tilde{i}_2 \\
Y_{aa} \mathbf{v}_a = I_\alpha + \tilde{i}_\alpha \\
\end{align*}
\]

(c) removed network

\[
Z_{\psi} \mathbf{i}_\psi = \tilde{e}_\psi
\]

Fig. 5.2
voltages $v_a$ consisting of $v_a, v_b, ..., v_e$ are the unknowns. In addition consider also the branch currents $i_\psi$ of the proposed removed branches shown by the broken lines, consisting of $i_j$ and $i_k$, as further unknowns. For the present it is assumed that there are no current or voltage sources in the removed branches. The problem is to obtain a set of equations from which, by elimination of the additional unknowns $i_\psi$, a solution for the unknown nodal voltages $v_a$ can be obtained.

After assigning arbitrary direction to the two branches shown by broken lines in fig. 5.2(a) and removing them, the original network is subdivided into two independent parts as shown in fig. 5.2(b) by separating at the common reference nodes. All the parts together with the equivalent current sources $i_\sim$, as defined below, are referred to as the equivalent network and shown in fig. 5.2(b). All the removed branches together with the equivalent voltage sources $e_\sim$, also defined below, are collectively referred to as the removed network and shown in fig. 5.2(c). The above definition is complementary to the one for mesh current analysis given on p. 132.

The two removed branches can be represented by equivalent hypothetical current sources $i_\sim$ consisting of $i_a, i_c, i_d, i_e$, flowing into nodes, of such magnitude and sense that all the nodal voltages in the equivalent network are the same as the corresponding nodal voltages in the original network. In a complementary manner the equivalent network can be represented by hypothetical voltage sources $e_\sim$, consisting of $e_j, e_k$, of such magnitude and sense that all the branch currents in the removed network are the same as the corresponding branch currents in the original network.

By inspection of fig. 5.2, it can be seen that the magnitude of the individual elements of the hypothetical current sources $i_\sim$ flowing into the nodes of the equivalent network, are the same as the sum of the currents in the removed branches which were connected to the corresponding nodes. (In fig. 5.2(b) there is only one branch removed from any node. Therefore in the particular example there is only one term in each sum of $i_\sim$).

Expressing the hypothetical current sources in terms of the defined mesh currents and hence in terms of the currents in the removed branches, for the particular example as shown in fig. 5.2, yields the equations
Basic Equations of Diakoptics

\[ \tilde{i}_a = i_j, \quad \tilde{i}_c = -i_k, \]
\[ \tilde{i}_d = i_k, \quad \tilde{i}_e = -i_j. \]

In the particular example no branches connected to node \( b \) have been removed so there is no need to add a current source to that node. In other words, the current source \( \tilde{i}_b \) is numerically equal to zero. The relationship between current sources at every node representing the removed branches and the assumed branch currents in all the removed branches may be expressed in matrix form as

\[
\begin{bmatrix}
\tilde{i}_a \\
\tilde{i}_b \\
\tilde{i}_c \\
\tilde{i}_d \\
\tilde{i}_e
\end{bmatrix} = \begin{bmatrix}
1 & . & . \\
. & . & . \\
. & . & -1 \\
. & . & 1 \\
-1 & . & .
\end{bmatrix} \begin{bmatrix}
i_j \\
i_k
\end{bmatrix}
\]

or in general terms as

\[ \tilde{i}_a = C_{\alpha \beta} i_\beta \] (5.9)

For any one removed branch, say branch \( j \), the corresponding column of connection matrix \( C_{\alpha \beta} \) may be constructed by inspection of fig. 5.2(a). The element \( C_{aj} \) in the row \( a \) and column \( j \) may be defined as

\[ C_{aj} = \begin{cases} 
+1, & \text{if the branch } j \text{ is directed towards node } a. \\
-1, & \text{if the branch } j \text{ is directed away from node } a. \\
0, & \text{if the node } a \text{ does not include branch } j.
\end{cases} \]

Similarly, it can be seen from fig. 5.2 that the magnitude of the hypothetical voltage sources \( \tilde{e}_\beta \), representing the component networks, are the same as the voltages across the removed branches. Expressing the voltage sources in terms of nodal voltages gives

\[ \tilde{e}_j = v_e - v_a \]
\[ \tilde{e}_k = v_c - v_d \]

In matrix notation these equations may be written as
\[
\begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k
\end{bmatrix} = \begin{bmatrix}
-1 & . & . & 1 \\
. & . & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e
\end{bmatrix}
\]

or in general terms as

\[
\tilde{e}_\psi = B_{\psi a} v_a
\] (5.10)

In a complementary manner, for the removed branch \( j \), the corresponding row of the matrix \( B_{\psi a} \) may also be constructed by inspection of fig. 5.2(a). The element \( B_{ja} \) in the row \( j \) and the column \( a \) may be defined as

\[
B_{ja} = \begin{cases} 
-1, & \text{if the branch } j \text{ is directed towards node } a. \\
+1, & \text{if the branch } j \text{ is directed away from node } a. \\
0, & \text{if the node } a \text{ does not include branch } j.
\end{cases}
\]

It can be seen from the definition of the connection matrix \( C_{\alpha \psi} \) and \( B_{\psi a} \) that the two matrices are simply related as

\[
B_{\psi a} = -C_{t\psi a}^t
\]

Therefore equation (5.10) may be stated as

\[
\tilde{e}_\psi = -C_{t\psi a} v_a
\] (5.11)

The nodal voltage equations which describe the equivalent network shown in fig. 5.2(b) written in matrix form are

\[
\begin{bmatrix}
(Y_1 + Y_3) & -Y_1 & 0 \\
-Y_1 & (Y_1 + Y_2 + Y_4) & -Y_2 \\
0 & -Y_2 & (Y_2 + Y_5)
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix} = \begin{bmatrix}
I_a + \tilde{I}_a \\
I_b + 0 \\
I_c + \tilde{I}_c
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
(Y_6 + Y_7) & -Y_6 \\
-Y_6 & (Y_6 + Y_8)
\end{bmatrix}
\begin{bmatrix}
v_d \\
v_e
\end{bmatrix} = \begin{bmatrix}
I_d + \tilde{I}_d \\
I_e + \tilde{I}_e
\end{bmatrix}
\]

The above equations may be written in compound matrix form as
Basic Equations of Diakoptics

\[
\begin{bmatrix}
\ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
l_1 + i_1^- \\
l_2 + i_2^- \\
\end{bmatrix}
\]

which can be written in general form as

\[
\tilde{Y}_{aa} v_a = l_a + i_a^-
\]  

(5.12)

The admittance matrix \(\tilde{Y}_{aa}\) of the equivalent network is in a block diagonal form. For computation purposes it is not necessary to store the whole matrix but only the diagonal blocks corresponding to the individual component networks. If two or more coefficient admittance matrices of the component networks are identical, it is necessary to store only one of them; the rest can be omitted. The corresponding matrix \(Y_{aa}\) of the original network is of the same order but is not in a block diagonal form and contains additional non-zero terms.

The equations relating the currents and voltages of the removed network as shown in fig. 5.2(c) can be written in matrix form as

\[
\begin{bmatrix}
Z_j & \cdot \\
\cdot & Z_k \\
\end{bmatrix}
\begin{bmatrix}
i_j \\
i_k \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{e}_j^- \\
\tilde{e}_k^- \\
\end{bmatrix}
\]

or in general form as

\[
Z_{\psi|\psi} i_{\psi} = \tilde{e}_{\psi}
\]

Substituting for \(\tilde{e}_{\psi}\) in the above equation from equation (5.11) and for \(i_{\alpha}\) in equation (5.12) from equation (5.9) results in

\[
Z_{\psi|\psi} i_{\psi} = -C_{\psi|\alpha}^t v_{\alpha}
\]

(5.13)

\[
\tilde{Y}_{aa} v_a = l_a + C_{\alpha|\psi} i_{\psi}
\]

(5.14)

The above equations can be stated in compound matrix form as

\[
\begin{bmatrix}
Z_{\psi|\psi} & C_{\psi|\alpha}^t \\
-C_{\alpha|\psi} & \tilde{Y}_{aa} \\
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
i_{\psi} \\
v_{\alpha} \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
l_a \\
\end{bmatrix}
\]

The solution for the unknown nodal voltages \(v_{\alpha}\) can be obtained from the above equation by eliminating the additional unknown mesh currents \(i_{\psi}\) and is shown in the following section. This is a complementary form of the fundamental equations of diakoptics (see equations 5.5 and 5.6).
Equations of solution

In a dual manner to that used for mesh current analysis, the equations of solution for the unknown nodal voltages \( \mathbf{v}_a \) may be obtained from the fundamental equations of diakoptics (5.13) and (5.14) by eliminating the additional unknown currents \( i_\psi \) in the following manner.

From equation (5.14)

\[
\mathbf{v}_a = \tilde{Y}^{-1}_{aa} (\mathbf{l}_a + \mathbf{C}_{a\psi} i_\psi)
\]  

(5.15)

Since \( \tilde{Y}_{aa} \) is a block diagonal matrix, \( \tilde{Y}^{-1}_{aa} \) is also a block diagonal matrix obtained by inverting the individual diagonal blocks of \( \tilde{Y}_{aa} \).

Substituting for \( \mathbf{v}_a \) into equation (5.13) results in

\[
(Z_{\psi\psi} + \mathbf{C}_{\psi a} \tilde{Y}^{-1}_{aa} \mathbf{C}_{a\psi}) i_\psi + \mathbf{C}_{\psi a} \tilde{Y}^{-1}_{aa} \mathbf{l}_a = 0
\]

Let

\[
\tilde{Z}_{\psi\psi} = Z_{\psi\psi} + \mathbf{C}_{\psi a} \tilde{Y}^{-1}_{aa} \mathbf{C}_{a\psi}
\]

then

\[
i_\psi = -\tilde{Z}^{-1}_{\psi\psi} \mathbf{C}_{\psi a} \tilde{Y}^{-1}_{aa} \mathbf{l}_a
\]

Substituting for \( i_\psi \) from the above equation into equation (5.15), the final equation of solution is obtained as

\[
\mathbf{v}_a = \tilde{Y}^{-1}_{aa} \mathbf{l}_a - \tilde{Y}^{-1}_{aa} \mathbf{C}_{a\psi} \tilde{Z}^{-1}_{\psi\psi} \mathbf{C}_{\psi a} \tilde{Y}^{-1}_{aa} \mathbf{l}_a
\]  

(5.16)

A more obvious method of solving equations (5.13) and (5.14) is by direct elimination of \( i_\psi \) to give

\[
\mathbf{v}_a = (\tilde{Y}_{aa} + \mathbf{C}_{a\psi} \tilde{Z}^{-1}_{\psi\psi} \mathbf{C}_{\psi a})^{-1} \mathbf{l}_a
\]

The above equation requires the evaluation of the inverse of \( \mathbf{Y}_{aa} \), the full true matrix of the original network. This is not in block diagonal form, i.e. it includes off-diagonal terms. Equation (5.16), by taking advantage of the block diagonal form of the coefficient matrix \( \tilde{Y}_{aa} \), is more economical in storage space and computation time. The true inverse matrix, if required, can always be obtained from equation (5.16) by replacing \( \mathbf{l}_a \) by a unit matrix.

Process of solution

The basic equations of diakoptics for both mesh and nodal systems have been derived by removing a number of individual branches which do not contain voltage or current sources,
resulting in a number of independent component networks. The removed branches must not form a closed loop and they must not contain nodes not included in the remaining network. The equations consist of as many variables as were in the original problem plus as many additional variables as there are removed branches.

The amount of calculation can be minimized if advantage is taken of the topological features of the network in selecting branches for removal. Ideal branches are those which separate subnetworks containing relatively large groups of meshes or nodes as the case may be. This is illustrated in the example on short circuit calculation shown on p. 153.

For most practical problems the simple form of the fundamental equations more than compensates for the additional complication involved in dealing with the new variables. For the mesh current analysis this simple form is

\[
\begin{bmatrix}
Y_{\psi\psi} & -C_{\psi\beta} \\
\tilde{Z}_{11} & \cdots \\
C_{\beta\psi}^t & \tilde{Z}_{ii} & \cdots \\
0 & \cdots & \tilde{Z}_{mm}
\end{bmatrix}
\begin{bmatrix}
v_{\psi} \\
\cdots \\
i_{\beta} \\
\cdots
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\cdots \\
E_{\beta}
\end{bmatrix}
\]

where \(\tilde{Z}_{ii}\) denotes the coefficient impedance matrix of \(\tilde{Z}_{\beta\beta}\) corresponding to the i-th component network and \(m\) is the total number of independent mesh component networks.

The solution may be obtained by partitioning the admittance matrix from the impedance matrices as previously shown and eliminating \(v_{\psi}\), thus

\[
v_{\psi} = \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} E_{\beta}
\]

Hence

\[
i_{\beta} = \tilde{Z}_{\beta\beta}^{-1} (E_{\beta} - C_{\beta\psi}^t v_{\psi})
\]

or

\[
i_{\beta} = (\tilde{Z}_{\beta\beta}^{-1} - \tilde{Z}_{\beta\beta}^{-1} C_{\beta\psi}^t \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1}) E_{\beta}
\]

where

\[
\tilde{Y}_{\psi\psi} = Y_{\psi\psi} + C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} C_{\beta\psi}^t
\]

In the nodal voltage analysis the fundamental equations of
diakoptics are

\[
\begin{bmatrix}
\mathbf{Z}_{\psi \psi} & \mathbf{C}_{\psi \alpha}^t \\
-\mathbf{C}_{\alpha \psi} & \tilde{\mathbf{Y}}_{11} & \cdots & \tilde{\mathbf{Y}}_{1n} \\
\vdots & & \ddots & \vdots \\
-\mathbf{C}_{\alpha \psi} & \cdots & \cdots & \tilde{\mathbf{Y}}_{nn}
\end{bmatrix}
\begin{bmatrix}
in_{\psi} \\
v_{\alpha}
\end{bmatrix}
= \begin{bmatrix}
0 \\
\cdots \\
\cdots \\
1_{\alpha}
\end{bmatrix}
\]

where \(\tilde{\mathbf{Y}}_{jj}\) denotes the coefficient admittance matrix of \(\tilde{\mathbf{Y}}_{aa}\) corresponding to the \(j\)-th component network and \(n\) is the total number of independent nodal component networks.

The solution may be obtained by partitioning the admittance matrices from the impedance matrix as previously shown and eliminating \(\mathbf{i}_{\psi}\), thus

\[
\mathbf{i}_{\psi} = -\tilde{\mathbf{Z}}_{\psi \psi}^{-1} \mathbf{C}_{\psi \alpha}^t \tilde{\mathbf{Y}}_{aa}^{-1} \mathbf{l}_{\alpha}
\]

Hence

\[
v_{\alpha} = \tilde{\mathbf{Y}}_{aa}^{-1} (\mathbf{l}_{\alpha} + \mathbf{C}_{\alpha \psi} \mathbf{i}_{\psi})
\] (5.15)

or

\[
v_{\alpha} = (\tilde{\mathbf{Y}}_{aa}^{-1} - \tilde{\mathbf{Y}}_{aa}^{-1} \mathbf{C}_{\alpha \psi} \tilde{\mathbf{Z}}_{\psi \psi}^{-1} \mathbf{C}_{\psi \alpha}^t \tilde{\mathbf{Y}}_{aa}^{-1}) \mathbf{l}_{\alpha}
\] (5.16)

where

\[
\tilde{\mathbf{Z}}_{\psi \psi} = \mathbf{Z}_{\psi \psi} + \mathbf{C}_{\psi \alpha}^t \tilde{\mathbf{Y}}_{aa}^{-1} \mathbf{C}_{\alpha \psi}
\]

For computational purposes it is not necessary to store the whole network matrix but only the diagonal blocks corresponding to the individual component networks. If two or more of the diagonal blocks are identical, it is necessary to store only one of them, thereby reducing the amount of calculation as well as storage.

It is shown in chapter 8 that, by considering subdivision in a more general manner, the removed network may include energy sources, closed loops and nodes not included in the remaining network. This sometimes results in the reduction of the original number of variables.

**Steps of solution**

Taking advantage of the block diagonal form of the coefficient matrix of the fundamental equations of diakoptics, the equations of solution may be obtained in simple steps as follows:

(i) Subdivide the original network into a number of component networks by removing selected branches.
(ii) Establish the appropriate connection matrix \( C_{\psi\beta} \) or \( C_{\alpha\psi} \).

(iii) Obtain the solution for the unknown mesh currents \( i_\beta \) or nodal voltages \( v_\alpha \) from equation (5.8) or (5.16).

\[
i_\beta = (\tilde{Z}_{\beta\beta}^{-1} - \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta}^t \tilde{Y}_{\psi\psi}^{-1} C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1}) E_\beta \quad (5.8)
\]

where \( \tilde{Y}_{\psi\psi} = Y_{\psi\psi} + C_{\psi\beta} \tilde{Z}_{\beta\beta}^{-1} C_{\psi\beta}^t \)

or \( v_\alpha = (\tilde{Y}_{\alpha\alpha}^{-1} - \tilde{Y}_{\alpha\alpha}^{-1} C_{\alpha\psi} \tilde{Z}_{\psi\psi}^{-1} C_{\psi\alpha} \tilde{Y}_{\alpha\alpha}^{-1}) I_\alpha \quad (5.16) \)

where \( \tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi\alpha}^t \tilde{Y}_{\alpha\alpha}^{-1} C_{\alpha\psi} \)

The following steps may then be used to obtain the required solution.

**Mesh analysis**

1. \( \tilde{Y}_{\beta\beta} = \tilde{Z}_{\beta\beta}^{-1} \)
2. \( \tilde{Y}_{\psi\psi} = Y_{\psi\psi} + C_{\psi\beta} \tilde{Y}_{\beta\beta}^{-1} C_{\beta\psi} \)
3. \( \tilde{Z}_{\psi\psi} = \tilde{Y}_{\psi\psi}^{-1} \)
4. \( i_\beta = \tilde{Y}_{\beta\beta} E_\beta \)
5. \( i_\beta' = C_{\psi\beta} i_\beta' \)
6. \( v_\psi = Z_{\psi\psi} i_\psi \)
7. \( i_\beta' = C_{\psi\beta} v_\psi \)
8. \( i_\beta'' = \tilde{Y}_{\beta\beta} v_\beta \)
9. \( i_\beta = i_\beta' - i_\beta'' \)

**Nodal analysis**

1. \( \tilde{Z}_{\alpha\alpha} = \tilde{Y}_{\alpha\alpha}^{-1} \)
2. \( \tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi\alpha}^t \tilde{Z}_{\alpha\alpha} C_{\alpha\psi} \)
3. \( v_\alpha' = \tilde{Z}_{\psi\psi}^{-1} \)
4. \( v_\alpha' = C_{\psi\alpha}^t v_\alpha' \)
5. \( i_\psi = Y_{\psi\psi} v_\psi \)
6. \( i_\alpha = C_{\alpha\psi} i_\psi \)
7. \( v_\alpha'' = \tilde{Z}_{\alpha\alpha} i_\alpha' \)
8. \( v_\alpha = v_\alpha' - v_\alpha'' \)

Note that from step (4) onwards the intermediate results are only vectors and that step (6) gives the voltages across the removed branches or the currents (with opposite sign) in them as the case may be. Also, because the connection matrices \( C_{\alpha\psi} \) and \( C_{\psi\beta} \) contain few non-zero terms and because these are either +1 or −1, any multiplication involving \( C_{\alpha\psi} \) or \( C_{\psi\beta} \) resolves itself to simple algebraic summation. For example, \( \tilde{Y}_{\psi\psi} \) or \( \tilde{Z}_{\psi\psi} \) of step (2) can be established by inspection as shown in chapter 7.

**Numerical examples**

As the first example consider the network shown in fig. 5.3(a) where all the branches have impedances of 1 ohm. Removing the
Fig. 5.3
two branches shown by broken lines, the equivalent network consisting of two parts can be constructed as shown in fig. 5.3(b). The various matrices of the equivalent and removed networks shown in fig. 5.3(c) can be constructed by inspection as

\[ \mathbf{Z}_{\beta\beta} = \begin{bmatrix}
3 & -1 & -1 & \cdots & .
-1 & 3 & -1 & \cdots & .
-1 & -1 & 2 & \cdots & .
. & . & . & 3 & -1 & -1
. & . & . & -1 & 3 & -1
. & . & . & -1 & -1 & 2
\end{bmatrix} \]

\[ \mathbf{i}_\beta = \begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
i_t \\
i_u \\
\end{bmatrix}, \quad \mathbf{E}_\beta = \begin{bmatrix}
14 \\
5 \\
-2 \\
5 \\
2 \\
-8 \\
\end{bmatrix}, \]

\[ \mathbf{Z}_{\psi\psi} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \mathbf{Y}_{\psi\psi} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \tilde{i}_\psi = \begin{bmatrix}
\tilde{i}_j \\
\tilde{i}_k \\
\end{bmatrix}, \quad \mathbf{v}_\psi = \begin{bmatrix}
v_j \\
v_k \\
\end{bmatrix} \]

Following the definition for the elements of the connection matrix \( \mathbf{C}_{\psi\beta} \) given on p. 134 results in

\[ \mathbf{C}_{\psi\beta} = \begin{bmatrix}
1 & . & . & . & . & -1 \\
. & . & -1 & 1 & . & . \\
\end{bmatrix} \]

The above connection matrix \( \mathbf{C}_{\psi\beta} \) could have been established from first principles by expressing the currents in the removed branches \( \tilde{i}_\psi \) in terms of the defined mesh currents \( i_\beta \) as shown on p. 134.

Following the steps outlined on p. 147 results in
\[
\begin{align*}
(1) \quad \tilde{\mathbf{Y}}_{\beta\beta} &= \frac{1}{8} \begin{bmatrix}
5 & 3 & 4 & . & . & . \\
3 & 5 & 4 & . & . & . \\
4 & 4 & 8 & . & . & . \\
. & . & . & 5 & 3 & 4 \\
. & . & . & 3 & 5 & 4 \\
. & . & . & 4 & 4 & 8 \\
\end{bmatrix} \\
(2) \quad \tilde{\mathbf{Y}}_{\psi\psi} &= \frac{1}{8} \begin{bmatrix}
21 & -8 \\
-8 & 21 \\
\end{bmatrix} \\
(3) \quad \mathbf{Z}'_{\psi\psi} &= \frac{8}{377} \begin{bmatrix}
21 & 8 \\
8 & 21 \\
\end{bmatrix} \\
(4) \quad \mathbf{i}'_{\beta} &= \frac{1}{8} \begin{bmatrix}
77 \\
59 \\
60 \\
-1 \\
-7 \\
-36 \\
\end{bmatrix} \\
(5) \quad \mathbf{i}'_{\psi} &= \frac{1}{8} \begin{bmatrix}
-113 \\
61 \\
\end{bmatrix} \\
(6) \quad \mathbf{v}_{\psi} &= \begin{bmatrix}
-5 \\
1 \\
\end{bmatrix} \\
(7) \quad \mathbf{v}'_{\beta} &= \begin{bmatrix}
-5 \\
0 \\
-1 \\
1 \\
0 \\
5 \\
\end{bmatrix} \\
(8) \quad \mathbf{i}''_{\beta} &= \frac{1}{8} \begin{bmatrix}
-29 \\
-19 \\
-28 \\
25 \\
23 \\
44 \\
\end{bmatrix} \\
(9) \quad \mathbf{i}_{\beta} &= \begin{bmatrix}
6 \\
5 \\
4 \\
3 \\
2 \\
1 \\
\end{bmatrix}
\end{align*}
\]

The reader can verify for himself that the solution satisfies the original problem.

As a second example consider the network shown in fig. 5.4(a) where all the branches have impedance of 1 ohm. Removing the two branches shown by broken lines, the equivalent network consisting of two parts can be constructed as shown in fig. 5.4(b).
(a) original network

(c) removed network

Fig. 5.4
The various matrices of the equivalent and removed networks shown in fig. 5.4(c) can be constructed by inspection as

\[
Y_{\alpha\alpha} = \begin{bmatrix}
2 & -1 & 0 & . & . \\
-1 & 3 & -1 & . & . \\
0 & -1 & 2 & . & . \\
. & . & . & 2 & -1 \\
. & . & . & -1 & 2
\end{bmatrix}
\quad
v_{\alpha} = \begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e
\end{bmatrix}
\quad
l_{\alpha} = \begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
Y_{\psi\psi} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad
Z_{\psi\psi} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad
i_{\psi} = \begin{bmatrix}
i_j \\
i_k
\end{bmatrix}
\quad
\tilde{e}_{\psi} = \begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k
\end{bmatrix}
\]

Following the definition for the elements of the connection matrix \( C_{\alpha\psi} \) given on p. 141 results in

\[
C_{\alpha\psi} = \begin{bmatrix}
1 & . \\
. & . \\
. & -1 \\
. & 1 \\
-1 & .
\end{bmatrix}
\]

The above connection matrix \( C_{\alpha\psi} \) could have been established from first principles by expressing the equivalent currents flowing into every node in terms of the currents flowing in the removed branches as shown on p. 141.

Following the steps outlined on p. 147 results in

\[
(1) \quad \tilde{Z}_{\alpha\alpha} = \frac{1}{24} \begin{bmatrix}
15 & 6 & 3 & . & . \\
6 & 12 & 6 & . & . \\
3 & 6 & 15 & . & . \\
. & . & 16 & 8 \\
. & . & 8 & 16
\end{bmatrix}
\]
Basic Equations of Diakoptics

\[ (2) \quad \hat{Z}^{\psi}_{\alpha} = \frac{1}{24} \begin{bmatrix} 55 & -11 \\ -11 & 55 \end{bmatrix} \]

\[ (3) \quad \mathbf{Y}^{\psi}_{\alpha} = \frac{1}{11} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \]

\[ (4) \quad \mathbf{v}^{\alpha} = \frac{1}{12} \begin{bmatrix} 9 \\ 6 \\ 9 \\ 4 \\ 8 \end{bmatrix} \]

\[ (5) \quad \mathbf{v}^{\psi} = \frac{1}{12} \begin{bmatrix} 1 \\ -5 \end{bmatrix} \]

\[ (6) \quad \mathbf{i}^{\psi} = \frac{1}{11} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \]

\[ (7) \quad i^{\alpha} = \frac{1}{11} \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} \]

\[ (8) \quad \mathbf{v}^{\alpha} = \frac{1}{132} \begin{bmatrix} 3 \\ 6 \\ 15 \\ -16 \\ -8 \end{bmatrix} \]

\[ (9) \quad \mathbf{v}^{\alpha} = \frac{1}{11} \begin{bmatrix} 8 \\ 5 \\ 7 \\ 5 \\ 8 \end{bmatrix} \]

The reader can verify for himself that the solution satisfies the original problem

**Three-phase short-circuit calculations**

The analysis of power system networks is essentially an analysis of electrical networks of the type dealt with in this chapter. The representation of the elements of power systems by equivalent networks is a very extensive subject [19] and beyond the scope of this book. In general, power system problems may be reduced to a nodal network and often, because of the geographical configuration, can be solved more rapidly by the method of diakoptics. A typical example of this is the three-phase short-circuit calculation.

The calculation of three-phase short-circuit currents involves the solution of the nodal network with the appropriate node short-circuited to common reference [20]. To determine short-circuit levels at different locations it is necessary to solve the network a number of times with various nodes short-circuited in turn.

The solution to the problem may be obtained either by an iterative approach or by a direct method. The iterative solution
requires the solution of the entire network for each fault condition. The direct method requires the formulation of the nodal admittance matrix \( Y \) before any faults have been applied and the evaluation of its inverse \( Z \), where \( Z = Y^{-1} \). The short-circuit information may then be obtained very quickly for all the different fault conditions.

The equations used in the three-phase fault calculations are

\[
\begin{align*}
Z &= Y^{-1} \\
\mathbf{v} &= Zl \\
I_a &= \frac{v_a}{Z_{aa}} \\
v_p^f &= v_p - I_a Z_{ap}
\end{align*}
\]

where

\[
\begin{align*}
Y &= \text{nodal admittance matrix of the network} \\
Z &= \text{inverse of the admittance matrix} \\
l &= \text{nodal injected currents} \\
\mathbf{v} &= \text{pre-fault voltages} \\
v_a &= \text{pre-fault voltage at the node } a \\
Z_{aa} &= \text{diagonal element of } Z \text{ in the row } a \text{ and column } a \\
&\quad \text{(short-circuit driving point impedance)} \\
I_a &= \text{fault current from the node } a \text{ to reference} \\
Z_{ap} &= \text{element of } Z \text{ in the row } a \text{ and column } p \\
v_p &= \text{pre-fault voltage at the node } p \\
v_p^f &= \text{voltage at the node } p \text{ with fault in the network}
\end{align*}
\]

For the particular application to the three-phase short-circuit calculation, the pre-fault voltages \( \mathbf{v} \) may be obtained with fewer calculations and may require less computer time if the original network is divided into a number of parts and equation (5.16) used

\[
\begin{align*}
v_a &= (Y_{aa}^{-1} - Y_{aa}^{-1} C_{\psi} Z_{\psi \psi}^{-1} C_{\psi a}^t Y_{aa}^{-1}) I_a \\
&= (Y_{aa}^{-1} - Y_{aa}^{-1} Z_{\psi \psi}^{-1} C_{\psi a}^t Y_{aa}^{-1}) I_a
\end{align*}
\]
where \( \bar{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi\alpha} Y_{\alpha\alpha}^{-1} C_{\alpha\psi} \)

and \( \bar{Y}_{\alpha\alpha} = \) admittance matrix in block diagonal form

\( C_{\alpha\psi} = \) connection matrix

\( I_{\alpha} = \) equivalent nodal injected currents

\( v_{\alpha} = \) pre-fault voltages

\( Z_{\psi\psi} = \) impedance matrix of removed branches

The evaluation of fault current and voltage distribution for a fault at any one node requires the elements of the true inverse matrix \( Z \) in the column corresponding to the number of the short-circuited node. The required elements may be obtained from equation (5.16) by simply setting the nodal currents \( I_{\alpha} \) to zero, except for the element in the row corresponding to the number of the node which is short-circuited. This element is set to unity.

\[
\begin{bmatrix}
Z_{1\alpha} \\
\vdots \\
Z_{a\alpha} \\
\vdots \\
Z_{p\alpha}
\end{bmatrix}
= (\bar{Y}_{\alpha\alpha}^{-1} - \bar{Y}_{\alpha\alpha}^{-1} C_{\alpha\psi} \bar{Z}_{\psi\psi}^{-1} C_{\psi\alpha} \bar{Y}_{\alpha\alpha}^{-1})
\begin{bmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{bmatrix}
\]

Power system networks are in general characterized by groups of closely interconnected nodes, for example industrial centres with concentrated distribution networks, interconnected through relatively few extra high-voltage circuits. Such networks can be solved to great advantage using the method of diakoptics. In practice each problem and the associated network is different from any other, but most problems fall into the general category described above and appropriate advantage can be taken of this fact in choosing the branches to be removed.

A digital computer and general program have been used to solve a number of power system networks and the practical advantages of diakoptics as applied to three-phase short-circuit calculations are illustrated by the graphs shown in fig. 5.5 to fig. 5.7.
The circuits investigated were typical power system networks with the number of 'cut' lines approximately equal to the number of component networks, each component network containing approximately the same number of nodes.

From Fig. 5.5 it can be seen that the required computer storage space is approximately proportional to $1/n$ where $n$ is the number of component networks. Fig. 5.6 shows that the matrix inversion time is proportional to $1/n^2$. The time required to formulate the equations by the computer is not dependent on the number of sub-divisions and is small compared with the time required for matrix inversion. Fig. 5.7 shows the total time and hence the relative cost of computation for a different number of fault
calculations on a 90-busbar network.

Fig. 5.6

It can be seen from fig. 5.7 that the calculation time and the cost of analysis are rapidly decreased by subdividing the system into a number of component networks. It must be appreciated, however, that as the network is subdivided into more parts the number of removed branches is also increased. By subdividing a given system into too many parts, the additional calculation created by the removed branches may outweigh the advantage of dealing with the system in parts. The break-even point varies with the type of problem.
Fig. 5.7

Problems

Fig. 5.8
1. By removing the branches shown by broken lines determine the mesh currents for the network shown in fig. 5.8 with

(a) \( Z_1 = 1 \text{ ohm}, \quad Z_2 = 1 \text{ ohm}. \)
(b) \( Z_1 = 2 \text{ ohms}, \quad Z_2 = 1 \text{ ohm}. \)
(c) \( Z_1 = 2 \text{ ohms}, \quad Z_2 = 2 \text{ ohms}. \)

2. By removing the branches shown by broken lines determine the nodal voltages for the network shown in fig. 5.9 with

(a) \( Y_1 = 1 \text{ mho}, \quad Y_2 = 1 \text{ mho}. \)
(b) \( Y_1 = 2 \text{ mhos}, \quad Y_2 = 1 \text{ mho}. \)
(c) \( Y_1 = 2 \text{ mhos}, \quad Y_2 = 2 \text{ mhos}. \)
CHAPTER SIX
Analysis of General Networks

Component networks without common reference point
It has been assumed so far that each component network contains a common reference node to which there are connections from one or more nodes in the individual component networks. It follows automatically that the admittance matrix of each subdivision can be inverted.

There are many problems in which it is not convenient to have a common reference in each component network. Such problems are more frequently encountered in networks which represent mechanical problems.

There are a number of ways of dealing with the problem. Essentially, it is necessary to select one node as a temporary reference in each component network which does not have a common reference node. The equations can then be established for each component network with respect to its own reference and therefore the resulting coefficient matrices can be inverted. The equations so established will not include the voltages of the temporary reference nodes with respect to the common or 'absolute' reference point for the system.

Since the total number of equations is reduced by one for every temporary reference node and the number of variables remains unchanged, it is necessary to introduce new equations before contemplating any solution. Such new equations can be established from the fact that in any one component network without the common reference, the algebraic sum of all current sources and all currents in the removed branches connected to that network must be equal to zero.

Functional equations
Consider the network shown in fig.6.1(a) where the current sources $I_a$ and $I_\delta$, consisting of $I_a$, $I_b$, ..., $I_f$ and $I_\alpha$, $I_h$, respectively, are known and the nodal voltages $v_a$ and $v_\delta$, consisting of $v_a$, $v_b$, ..., $v_f$ and $v_\alpha$, $v_h$, respectively, are unknown.

Dividing the network into three parts and selecting in two of the subdivisions a temporary reference node, the equivalent
(a) original network \( \mathbf{Y}_a \mathbf{v}_a = \mathbf{l}_a \)

(b) equivalent network \( \tilde{\mathbf{Y}}_a \mathbf{v}_a = \mathbf{l}_a + \tilde{\mathbf{l}}_a \)

(c) removed network \( \mathbf{Z}_\psi \mathbf{i}_\psi = \tilde{\mathbf{e}}_\psi \)

Fig. 6.1
network as shown in fig. 6.1(b) can be constructed. The subscript \( \delta \) refers to quantities associated with temporary reference node numbers shown in fig. 6.1 by shaded squares g and h. All the unknown voltages \( v_a \) and \( v_\delta \) are expressed with respect to the common reference point as shown. In the equivalent network, the nodal voltages \( v'_a \) consisting of \( v'_a, v'_b, \ldots, v'_f \) are expressed with respect to the individual reference nodes g, h and the common reference point as appropriate.

In chapter 5 it was shown that the hypothetical current sources flowing into the various nodes of the equivalent networks may be expressed in terms of the currents flowing in the removed branches. For the particular example, by inspection of fig. 6.1(a) and 6.1(b),

\[
\begin{bmatrix}
\tilde{i}_a \\
\tilde{i}_b \\
\tilde{i}_c \\
\tilde{i}_d \\
\tilde{i}_e \\
\tilde{i}_f \\
\tilde{i}_g \\
\tilde{i}_h
\end{bmatrix} =
\begin{bmatrix}
. & . & . & & & \tilde{i}_j \\
. & -1 & . & & & \tilde{i}_k \\
1 & . & . & & & \tilde{i}_l \\
. & 1 & -1 & & & \\
. & . & . & & & \\
. & . & 1 & & & \\
. & . & . & . & & \\
-1 & . & . & & & \\
\end{bmatrix}
\]

Partitioning the equations relating the currents flowing into the temporary reference nodes from the rest and expressing the above equations in a compound matrix form results in

\[
\begin{bmatrix}
\tilde{i}_a \\
\vdots \\
\tilde{i}_\delta \\
\tilde{i}_h
\end{bmatrix} =
\begin{bmatrix}
C_{a\,\psi} & \cdots \\
\cdots & \cdots \\
C_{\delta\,\psi} & \cdots \\
\end{bmatrix}
\begin{bmatrix}
i_\psi
\end{bmatrix}
\]

or

\[
\tilde{i}_a = C_{a\,\psi} i_\psi \\
\tilde{i}_\delta = C_{\delta\,\psi} i_\psi
\]  

Similarly, the voltage sources shown in fig. 6.1(c) may be expressed in terms of all the nodal voltages. For the particular example,
\[
\begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k \\
\tilde{e}_1
\end{bmatrix} =
\begin{bmatrix}
. & . & -1 & . & . & . & 1 \\
. & 1 & . & -1 & . & . & . \\
. & . & . & 1 & . & -1 & .
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f \\
v_g \\
v_h
\end{bmatrix}
\]

Partitioning in a complementary manner to the current relationships and using the topological considerations outlined in chapter 5 the above equations may be expressed in a compound matrix form as

\[
\begin{bmatrix}
\tilde{e}_\psi
\end{bmatrix} =
\begin{bmatrix}
-C_{\psi\alpha}^{t} & -C_{\psi\delta}^{t}
\end{bmatrix}
\begin{bmatrix}
v_\alpha \\
v_\delta
\end{bmatrix}
\]

or as

\[
\tilde{e}_\psi = -C_{\psi\alpha}^{t}v_\alpha - C_{\psi\delta}^{t}v_\delta \quad (6.3)
\]

From fig. 6.1(a) all the nodal voltages may be expressed in terms of \(v'_\alpha\) and \(v_\delta\) as follows

\[
\begin{align*}
v_a &= v'_a + v_g \\
v_b &= v'_b + v_g \\
v_c &= v'_c + v_h \\
v_d &= v'_d + v_h \\
v_e &= v'_e \\
v_f &= v'_f \\
v_g &= v_g \\
v_h &= v_h
\end{align*}
\]

which can be stated in matrix form as
\[
\begin{bmatrix}
  v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f \\
\cdots \\
v_g \\
v_h
\end{bmatrix} = \begin{bmatrix}
  v'_a \\
v'_b \\
v'_c \\
v'_d \\
v'_e \\
v'_f \\
\cdots \\
\cdot \\
\cdot
\end{bmatrix} + \begin{bmatrix}
  1 & . & . & . \\
  . & 1 & . & . \\
  . & . & 1 & . \\
  \cdots & \cdots & \cdots & \cdots \\
  . & . & . & 1
\end{bmatrix}
\begin{bmatrix}
v_g \\
v_h
\end{bmatrix}
\]

Partitioning the equations relating the voltages at the temporary nodes from the rest and expressing the above relations in a compound matrix form results in

\[
\begin{bmatrix}
  v_a \\
\cdots \\
v_\delta
\end{bmatrix} = \begin{bmatrix}
  v'_a \\
\cdots \\
  0
\end{bmatrix} + \begin{bmatrix}
  k_{a\delta} \\
\cdots \\
u_{\delta\delta}
\end{bmatrix}\begin{bmatrix}
v_\delta
\end{bmatrix}
\]

Hence

\[
v_a = v'_a + k_{a\delta} v_\delta
\]

(6.4)

where \( u_{\delta\delta} \) is a unit matrix and the element \( k_{ag} \) in row a corresponding to node a, and column \( g \) corresponding to temporary reference node \( g \) of the matrix \( k_{a\delta} \) may be defined as

\[
k_{ag} = \begin{cases}
  +1, & \text{if nodes } a \text{ and } g \text{ are in the same component network.} \\
  0, & \text{otherwise.}
\end{cases}
\]

Summing all nodal currents in each component network without the common reference node results, by inspection of fig. 6.1(b), in

\[
L_a + L_b + \tilde{i}_b + \tilde{i}_g + \tilde{i}_g = 0
\]

\[
L_c + \tilde{i}_c + L_d + \tilde{i}_d + L_h = 0
\]

Since \( \tilde{i}_a = \tilde{i}_h = 0 \), the above relationships may be expressed in matrix form as
\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\vdots & 1 & 1 & \ldots & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_a \\
\vdots \\
I_h \\
\end{bmatrix}
+ 
\begin{bmatrix}
\tilde{i}_a \\
\vdots \\
\tilde{i}_h \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

or in a compound matrix form as
\[
\begin{bmatrix}
L_{\delta a} & U_{\delta\delta} \\
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
I_a \\
\vdots \\
I_h \\
\end{bmatrix} \\
\begin{bmatrix}
\tilde{i}_a \\
\vdots \\
\tilde{i}_h \\
\end{bmatrix} \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

or in general terms as
\[
L_{\delta a} I_a + U_{\delta\delta} I_\delta + L_{\delta a} \tilde{i}_a + U_{\delta\delta} \tilde{i}_\delta = 0
\]

where \( U_{\delta\delta} \) is a unit matrix and the element \( L_{\delta a} \) in the row \( \gamma \) and column \( \alpha \) of the matrix \( L_{\delta a} \) may be defined as
\[
L_{\gamma\alpha} \equiv \begin{cases} 
+1, & \text{if nodes } \alpha \text{ and } \gamma \text{ are in the same component network.} \\
0, & \text{otherwise} 
\end{cases}
\]

From the above definition it can be seen that the matrix \( L_{\delta a} \) and \( K_{\alpha\delta} \) are simply related as
\[
L_{\delta a} = \left( K_{\delta a} \right)^t
\]

hence equation (6.5) may be stated as
\[
K_{\delta a}^t I_a + I_\delta + K_{\delta a}^t \tilde{i}_a + \tilde{i}_\delta = 0
\]

Substituting in the above equation for \( \tilde{i}_a \) and \( \tilde{i}_\delta \), where, from equations (6.1) and (6.2),
\[
\tilde{i}_a = C_{\alpha\psi} i_\psi \\
\tilde{i}_\delta = C_{\delta\psi} i_\psi
\]
results in,  
\[ K_{\delta \alpha}^t l_{\delta} + l_{\delta} + (K_{\delta \alpha}^t C_{\alpha \psi} + C_{\delta \psi}) i_{\psi} = 0. \]

Let
\[ M_{\delta \psi} = K_{\delta \alpha}^t C_{\alpha \psi} + C_{\delta \psi} \quad (6.6) \]

and
\[ J_{\delta} = K_{\delta \alpha}^t l_{\alpha} + l_{\delta} \]

Then the above equation may be stated as
\[ J_{\delta} + M_{\delta \psi} i_{\psi} = 0 \quad (6.7) \]

It can be seen that the elements of \( J_{\delta} \) represent the total transfer of current from the corresponding component network. In practice, any element \( J_g \) corresponding to the component network containing temporary reference node \( g \) can be evaluated by inspection of the equivalent network. It consists of the algebraic sum of all nodal currents in the component network including the nodal current at the node \( g \) but excluding all the hypothetical current sources \( \tilde{i}_\alpha \) representing the removed network.

The elements of the connection matrix \( M_{\delta \psi} \) may also be obtained by inspection. The element \( M_{g p} \) in the row \( g \) and column \( p \) may be defined as

\[ M_{g p} = \begin{cases} 
+1, & \text{if the current of removed branch } p \text{ flows into the network containing the temporary reference node } g. \\
-1, & \text{if the current of removed branch } p \text{ flows away from the network containing temporary reference node } g. \\
0, & \text{if the removed branch } p \text{ is not connected to the subnetwork containing the temporary reference node } g. 
\end{cases} \]

The equations relating the currents and voltages of the removed network shown in fig. 6.1(c) can be written in matrix form as

\[
\begin{bmatrix}
Z_j & . & . \\
. & Z_k & . \\
. & . & Z_1
\end{bmatrix}
\begin{bmatrix}
i_j \\
i_k \\
i_1
\end{bmatrix} =
\begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k \\
\tilde{e}_1
\end{bmatrix}
\]

or in general terms as
\[ Z_{\psi \psi} i_{\psi} = \tilde{e}_{\psi} \quad (6.8) \]

The nodal voltage equations which describe the equivalent network shown in fig. 6.1(b) written in matrix form are
\[ \begin{bmatrix} (Y_1 + Y_2) & -Y_2 & 0 \\ -Y_2 & (Y_2 + Y_3) & 0 \\ 0 & (Y_4 + Y_5) & -Y_5 \\ -Y_5 & (Y_5 + Y_6) & 0 \\ 0 & 0 & (Y_7 + Y_8) \\ -Y_8 & (Y_8 + Y_9) & 0 \end{bmatrix} \begin{bmatrix} v'_a \\ v'_b \\ v'_c \\ v'_d \\ v'_e \\ v'_f \end{bmatrix} = \begin{bmatrix} l_a + 0 \\ l_b + \tilde{i}_b \\ l_c + \tilde{i}_c \\ l_d + \tilde{i}_d \\ l_e + 0 \\ l_f + \tilde{i}_f \end{bmatrix} \]

Writing the above equation in a general form,

\[ \tilde{Y}_{aa} v'_a = l_a + \tilde{i}_a \quad (6.9) \]

Collecting together the relevant equations,

\[ \tilde{e}_\psi = -C^t_{\psi \alpha} v_\alpha - C^t_{\psi \delta} v_\delta \quad [(6.3)] \]

\[ v_\alpha = v'_\alpha + K_{\alpha \delta} v_\delta \quad [(6.4)] \]

\[ M_{\delta \psi} = K_{\delta \alpha} C_{\alpha \psi} + C_{\delta \psi} \quad [(6.6)] \]

Substituting in equation (6.3) for \( v_\alpha \) from equation (6.4) results in

\[ \tilde{e}_\psi = -C^t_{\psi \alpha} (v'_\alpha + K_{\alpha \delta} v_\delta) - C^t_{\psi \delta} v_\delta \]

therefore

\[ \tilde{e}_\psi = -C^t_{\psi \alpha} v'_\alpha - (C^t_{\psi \alpha} K_{\alpha \delta} + C^t_{\psi \delta}) v_\delta \]

but, from equation (6.6),

\[ M^t_{\psi \delta} = C^t_{\psi \alpha} K_{\alpha \delta} + C^t_{\psi \delta} \]

hence

\[ \tilde{e}_\psi = -C^t_{\psi \alpha} v'_\alpha - M^t_{\psi \delta} v_\delta \quad (6.10) \]

Collecting together the relevant equations

\[ \tilde{i}_\alpha = C_{\alpha \psi} i_\psi \quad [(6.1)] \]

\[ J_\delta + M_{\delta \psi} i_\psi = 0 \quad [(6.7)] \]

\[ Z_{\psi \psi} i_\psi = \tilde{e}_\psi \quad [(6.8)] \]

\[ \tilde{Y}_{aa} v'_a = l_a + \tilde{i}_a \quad [(6.9)] \]

\[ \tilde{e}_\psi = -C^t_{\psi \alpha} v'_\alpha - M^t_{\psi \delta} v_\delta \quad (6.10) \]

Substituting for \( \tilde{i}_\alpha \) in equation (6.9) from equation (6.1) and for \( \tilde{e}_\psi \) in equation (6.8) from equation (6.10) results in
\[ \tilde{Y}_{aa} v'_a = I_a + C_{a\psi} i_{\psi} \]  \hspace{1cm} (6.11)
\[ Z_{\psi\psi} i_{\psi} = -C_{\psi a} v'_a - M_{\psi\delta} v_{\delta} \]  \hspace{1cm} (6.12)

The above equations together with equation (6.7) may be stated in compound matrix form as

\[
\begin{bmatrix}
\tilde{Y}_{aa} & -C_{a\psi} & \cdots \\
C_{\psi a}^t & Z_{\psi\psi} & M_{\psi\delta}^t \\
\cdots & -M_{\delta\psi} & \cdots
\end{bmatrix}
\begin{bmatrix}
v'_a \\
i_{\psi} \\
v_{\delta}
\end{bmatrix}
= \begin{bmatrix}
I_a \\
0 \\
J_{\delta}
\end{bmatrix}
\]

The above relationships are the fundamental equations of diakoptics for the special case of subnetworks without a common reference point. The relationships are similar in form to equations (5.13) and (5.14) of chapter 5 but extended by one row and column.

**Equations of solution**

The solution for the unknown nodal voltages \( v_a \) may be obtained from equations (6.11), (6.12) and (6.7).

From equation (6.11),

\[ v'_a = \tilde{Y}_{aa}^{-1} (I_a + C_{a\psi} i_{\psi}) \]  \hspace{1cm} (6.13)

Substituting for \( v'_a \) from equation (6.13) into (6.12) results in

\[ (Z_{\psi\psi} + C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi}) i_{\psi} + C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a + M_{\psi\delta}^t v_{\delta} = 0 \]

Let

\[ \tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} \]  \hspace{1cm} (6.14)

then

\[ i_{\psi} = -\tilde{Z}_{\psi\psi}^{-1} (C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a + M_{\psi\delta}^t v_{\delta}) \]  \hspace{1cm} (6.15)

Multiplying both sides of the above equation by \(-M_{\delta\psi}\) and using equation (6.7),

\[ -M_{\delta\psi} i_{\psi} = M_{\delta\psi} \tilde{Z}_{\psi\psi}^{-1} (C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a + M_{\psi\delta}^t v_{\delta}) = J_{\delta} \]

or

\[ M_{\delta\psi} \tilde{Z}_{\psi\psi}^{-1} M_{\psi\delta} v_{\delta} = J_{\delta} - M_{\delta\psi} \tilde{Z}_{\psi\psi}^{-1} C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a . \]

Let

\[ Y_{\delta\delta} = M_{\delta\psi} \tilde{Z}_{\psi\psi}^{-1} M_{\psi\delta} \]  \hspace{1cm} (6.16)

then

\[ v_{\delta} = \tilde{Y}_{\delta\delta}^{-1} (J_{\delta} - M_{\delta\psi} \tilde{Z}_{\psi\psi}^{-1} C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a) \]  \hspace{1cm} (6.17)

The connection matrix \( M_{\delta\psi} \) can be obtained by inspection.
using the definition shown on p. 166 or from equation (6.6)

\[ M_\delta \psi = K_{\delta \alpha}^t C_{\alpha \psi} + C_{\delta \psi} \]  

[(6.6)]

The current \( i_\psi \) can be evaluated from equation (6.15) and the solution for \( v_\delta \)

\[ i_\psi = -\tilde{Z}_\psi^{-1}(C_{\psi \alpha} \tilde{Y}_{\alpha \alpha}^{-1} l_\alpha + M_{\psi \delta} v_\delta) \]  

[(6.15)]

The voltages \( v'_\alpha \) can be evaluated from equation (6.13) and the solution for \( i_\psi \)

\[ v'_\alpha = \tilde{Y}_{\alpha \alpha}^{-1} (l_\alpha + C_{\alpha \psi} i_\psi) \]  

[(6.13)]

The final nodal voltages \( v_\alpha \) can be evaluated from equation (6.4) and the solution for \( v'_\alpha \) and \( v_\delta \)

\[ v_\alpha = v'_\alpha + K_{\alpha \delta} v_\delta \]  

[(6.4)]

**Steps of solution**

The solution for \( v_\alpha \) may be obtained by evaluating equations (6.14), (6.17), (6.15), (6.13) and (6.4) in that order.

Taking advantage of the block diagonal form of the coefficient matrix of the fundamental equations of diakoptics, the equations of solution may be obtained in simple steps as follows.

(i) Subdivide the original network into a number of component networks by removing selected branches.

(ii) For the component networks which do not contain the common reference node, select arbitrarily one node as a temporary reference.

(iii) Establish:

1. \( \tilde{Y}_{\alpha \alpha} \) = admittance matrix of the individual component networks with respect to the common or the temporary reference node.

2. \( Z_{\psi \psi} \) = impedance matrix of the removed branches.

3. \( l_\alpha \) = nodal current vector of all the nodal currents except the currents at the temporary reference nodes.

4. \( J_\delta \) = total nodal current in the networks with the temporary reference node, (including the current at the temporary reference node).

5. \( C_{\alpha \psi}, K_{\alpha \delta}, M_{\delta \psi} \) = connection matrices
(iv) Evaluate:

(1) \( \tilde{Z}_{aa} = \tilde{Y}_{aa}^{-1} \)
(2) \( \tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi\alpha} Z_{aa} C_{\alpha\psi} \)
(3) \( Y_{\psi\psi} = \tilde{Z}_{\psi\psi}^{-1} \)
(4) \( Y_{\delta\delta} = M_{\delta\psi} Y_{\psi\psi} M_{\psi\delta}^t \)
(5) \( Z_{\delta\delta} = Y_{\delta\delta}^{-1} \)
(6) \( v_a'' = \tilde{Z}_{aa} l_a \)
(7) \( v_{\psi} = C_{\psi\alpha} v_{\alpha}'' \)
(8) \( i_{\psi}' = Y_{\psi\psi} v_{\psi} \)
(9) \( I_{\delta} = J_{\delta} - M_{\delta\psi} i_{\psi}' \)
(10) \( v_{\delta} = Z_{\delta\delta} l_{\delta} \)
(11) \( v_{\psi}' = v_{\psi} + M_{\psi\delta} v_{\delta} \)
(12) \( i_{\psi} = - Y_{\psi\psi} v_{\psi}' \)
(13) \( I_{\alpha}' = I_{\alpha} + C_{\alpha\psi} i_{\psi} \)
(14) \( v_{\alpha}' = \tilde{Z}_{\alpha\alpha} l_{\alpha}' \)
(15) \( v_{\alpha} = v_{\alpha}' + K_{\alpha\delta} v_{\delta} \)

**Numerical example**

As an example, consider the network shown in fig. 6.2(a) where all the branches have an impedance of 1 ohm. Removing the three branches shown by broken lines, the equivalent network consisting of three independent parts is constructed as shown in fig. 6.2(b). The various matrices of the equivalent and removed networks shown in fig. 6.2(c) can be constructed by inspection as

\[
\tilde{Y}_{aa} = \begin{bmatrix}
2 & -1 & . & . & . & . \\
-1 & 2 & . & . & . & . \\
. & . & 2 & -1 & . & . \\
. & . & -1 & 2 & . & . \\
. & . & . & 2 & -1 & . \\
. & . & . & . & -1 & 2 \\
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
a & 0 \\
b & -1 \\
c & -6 \\
d & -6 \\
e & 3 \\
f & -2 \\
\end{bmatrix}
\]
(a) original network

(b) equivalent network

(c) removed network

Fig. 6.2
\[
\mathbf{C}_{\alpha \psi} = \begin{bmatrix}
  a & . & . \\
  b & -1 & . \\
  c & 1 & . \\
  d & . & 1 \ -1 \\
  e & . & . \\
  f & . & 1
\end{bmatrix}, \\
\mathbf{K}_{\alpha \delta} = \begin{bmatrix}
  a & 1 \\
  b & 1 \\
  c & . & 1 \\
  d & . & 1 \\
  e & . & . \\
  f & . & .
\end{bmatrix}
\]

\[
\mathbf{Z}_{\psi \psi} = \begin{bmatrix}
  j & k & 1 \\
  j & 1 & . \\
  1 & . & 1
\end{bmatrix}, \\
\mathbf{J}_{\delta} = \begin{bmatrix}
  g & 5 \\
  h & -3
\end{bmatrix}
\]

\[
\mathbf{M}_{\delta \psi} = \begin{bmatrix}
  g & -1 & -1 & . \\
  h & 1 & 1 & -1
\end{bmatrix}
\]

(1) \( \tilde{\mathbf{Z}}_{\alpha \alpha} = \tilde{\mathbf{Y}}_{\alpha \alpha}^{-1} \)

\[
\tilde{\mathbf{Z}}_{\alpha \alpha} = \frac{1}{3} \begin{bmatrix}
  2 & 1 & . & . & . \\
  1 & 2 & . & . & . \\
  . & . & 2 & 1 & . \\
  . & . & 1 & 2 & . \\
  . & . & . & 2 & 1 \\
  . & . & . & 1 & 2
\end{bmatrix}
\]
(2) $\tilde{\mathbf{Z}}_{\psi\psi} = \mathbf{Z}_{\psi\psi} + \mathbf{C}_{\psi\alpha}' \tilde{\mathbf{Z}}_{\alpha\alpha} \mathbf{C}_{\alpha\psi}$

$\tilde{\mathbf{Z}}_{\psi\psi} = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 7 & -2 \\ -1 & -2 & 7 \end{bmatrix}$

$\mathbf{Y}_{\psi\psi} = \frac{3}{215} \begin{bmatrix} 45 & -5 & 5 \\ -5 & 34 & 9 \\ 5 & 9 & 34 \end{bmatrix}$

(3) $\mathbf{Y}_{\psi\psi} = \tilde{\mathbf{Z}}_{\psi\psi}^{-1}$

(4) $\mathbf{Y}_{\delta\delta} = \mathbf{M}_{\delta\psi} \mathbf{Y}_{\psi\psi} \mathbf{M}_{\psi\delta}^t$

$\mathbf{Y}_{\delta\delta} = \frac{3}{215} \begin{bmatrix} 69 & -55 \\ -55 & 75 \end{bmatrix}$

$\mathbf{z}_{\delta\delta} = \frac{1}{30} \begin{bmatrix} 75 & 55 \\ 55 & 69 \end{bmatrix}$

(5) $\mathbf{Z}_{\delta\delta} = \mathbf{Y}_{\delta\delta}^{-1}$

(6) $\mathbf{v}_{\alpha}'' = \tilde{\mathbf{Z}}_{\alpha\alpha} \mathbf{l}_\alpha$

$\mathbf{v}_{\alpha}'' = \frac{1}{3} \begin{bmatrix} -1 \\ -2 \\ -18 \\ -18 \\ 4 \\ -1 \end{bmatrix}$

(7) $\mathbf{v}_{\psi} = \mathbf{C}_{\psi\alpha}^t \mathbf{v}_{\alpha}''$

$\mathbf{v}_{\psi} = \frac{1}{3} \begin{bmatrix} -18 \\ -16 \\ 17 \end{bmatrix}$

(8) $\mathbf{i}_{\psi}' = \mathbf{Y}_{\psi\psi} \mathbf{v}_{\psi}$

$\mathbf{i}_{\psi}' = \frac{1}{3} \begin{bmatrix} -15 \\ -7 \\ 8 \end{bmatrix}$

(9) $\mathbf{l}_\delta = \mathbf{J}_\delta - \mathbf{M}_{\delta\psi} \mathbf{i}_{\psi}'$

$\mathbf{l}_\delta = \frac{1}{5} \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

(10) $\mathbf{v}_\delta = \mathbf{Z}_{\delta\delta} \mathbf{l}_\delta$

$\mathbf{v}_\delta = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

(11) $\mathbf{v}_\psi' = \mathbf{v}_{\psi} + \mathbf{M}_{\psi\delta}^t \mathbf{v}_{\delta}$

$\mathbf{v}_\psi' = \frac{1}{3} \begin{bmatrix} -15 \\ -13 \\ -7 \end{bmatrix}$

(12) $\mathbf{i}_\psi = -\mathbf{Y}_{\psi\psi} \mathbf{v}_\psi'$

$\mathbf{i}_\psi = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$
(13) \( I'_a = I_a + C_{a\psi} i_{\psi} \)

(14) \( v'_a = Z_{a\alpha} I'_a \)

(15) \( v_\alpha = v'_a + K_{a\delta} v_\delta \)

\[
I'_a = \begin{bmatrix}
0 \\
-3 \\
-3 \\
-6 \\
3 \\
0
\end{bmatrix}, \quad
v'_a = \begin{bmatrix}
-1 \\
-2 \\
-4 \\
-5 \\
2 \\
1
\end{bmatrix}, \quad
v_\alpha = \begin{bmatrix}
6 \\
5 \\
4 \\
3 \\
2 \\
1
\end{bmatrix}
\]

**Mixed networks**

General mixed networks are those which contain both current sources and voltage sources. Particular difficulties arise in nodal analysis if voltage sources are present in the removed branches. In that case the fundamental equations of diakoptics as developed in chapter 5 must be extended.

**Functional Equations**

As an example, consider the network shown in fig. 6.3(a) where both removed branches include a known series voltage source \( E_\psi \) consisting of \( E_j \) and \( E_k \). Let the current sources \( I_\alpha \) consisting of \( I_a, I_b, \ldots, I_f \) be also known and the nodal voltages \( v_\alpha \) consisting of \( v_a, v_b, \ldots, v_f \) be unknown.

The two proposed removed branches with voltage sources can be represented by equivalent hypothetical current sources \( \tilde{i}_a \), flowing into the nodes, of such magnitude and sense that all nodal voltages in the equivalent network shown in fig. 6.3(b) are the same as the corresponding nodal voltages in the original network. In a complementary manner the equivalent network can be represented by hypothetical voltage sources \( \tilde{e}_\psi \) of such magnitude and sense that all the branch currents in the removed network, shown in fig. 6.3(c), are the same as the corresponding branch currents in the original network.

Expressing the hypothetical current sources in terms of the defined mesh currents and hence in terms of the currents in the removed branches results in
(a) original network

(b) equivalent network

(c) removed network

Fig. 6.3
\[ \tilde{i}_a = -i_j \]
\[ \tilde{i}_c = -i_k \]
\[ \tilde{i}_d = i_j \]
\[ \tilde{i}_e = i_k \]

The relationship between current sources representing the removed branches and the branch currents in all the removed branches, may be expressed in matrix form as

\[
\begin{bmatrix}
\tilde{i}_a \\
\tilde{i}_b \\
\tilde{i}_c \\
\tilde{i}_d \\
\tilde{i}_e \\
\tilde{i}_f
\end{bmatrix} =
\begin{bmatrix}
-1 & . & . \\
. & . & -1 \\
. & 1 & . \\
. & . & 1
\end{bmatrix}
\begin{bmatrix}
i_j \\
i_k
\end{bmatrix}
\]

or in general terms as

\[ \tilde{i}_a = C_{a,\psi} \tilde{i}_\psi \]  

(6.18)

Similarly, it can be seen from fig. 6.3(c) that the magnitude of the hypothetical voltage sources \( \tilde{e}_\psi \) representing the component networks, are the same as the voltage across the removed branches, consisting of the voltage across the branch impedance and any voltage source in series with the branch impedance.

Expressing in matrix form the relationship between voltage sources \( \tilde{e}_\psi \) of every removed branch and the nodal voltages, results in

\[
\begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k
\end{bmatrix} =
\begin{bmatrix}
1 & . & . & -1 & . \\
. & 1 & . & -1
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f
\end{bmatrix}
\]
or in general terms

\[ \tilde{e}_\psi = B_{\psi \alpha} v_\alpha \]

The matrices \( C_{\alpha \beta} \) and \( B_{\psi \alpha} \) are defined in chapter 5 from which it follows that

\[ B_{\psi \alpha} = -C_{\psi \alpha}^t \]

therefore

\[ \tilde{e}_\psi = -C_{\psi \alpha}^t v_\alpha \]  \hspace{1cm} (6.19)

The equations relating the currents and voltages of the removed network, shown in fig. 6.3(c) can be written in matrix form as

\[
\begin{bmatrix}
Z_j & . \\
. & Z_k
\end{bmatrix}
\begin{bmatrix}
i_j \\
i_k
\end{bmatrix}
= \begin{bmatrix}
E_j + \tilde{e}_j \\
E_k + \tilde{e}_k
\end{bmatrix}
\]

or in general form as

\[ Z_{\psi \psi} i_\psi = E_{\psi} + \tilde{e}_\psi \]  \hspace{1cm} (6.20)

The nodal voltage equations which describe the equivalent network, shown in fig. 6.3(b), written in matrix form are

\[
\begin{bmatrix}
(Y_1 + Y_2) & -Y_1 & -Y_2 \\
-Y_1 & (Y_1 + Y_3 + Y_4) & -Y_3 \\
-Y_2 & -Y_3 & (Y_2 + Y_3 + Y_5)
\end{bmatrix}
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
(Y_6 + Y_7) & -Y_6 & -Y_7 \\
-Y_6 & (Y_6 + Y_8 + Y_9) & -Y_8 \\
-Y_7 & -Y_9 & (Y_7 + Y_8 + Y_9)
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f
\end{bmatrix}
\]

\[
\begin{bmatrix}
l_a + \tilde{i}_a \\
l_b + 0 \\
l_c + \tilde{i}_c \\
l_d + \tilde{i}_d \\
l_e + \tilde{i}_e \\
l_f + 0
\end{bmatrix}
= \begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f
\end{bmatrix}
\]
or in general terms
\[ \tilde{Y}_{aa} \mathbf{v}_a = \mathbf{l}_a + \tilde{i}_a. \]
Substituting for \( \tilde{i}_a \) in the above equation from equation (6.18) and for \( \tilde{e}_\psi \) in equation (6.20) from equation (6.19) results in
\[ \tilde{Y}_{aa} \mathbf{v}_a = \mathbf{l}_a + C_{a\psi} \mathbf{i}_\psi \]  \hspace{1cm} (6.21)
\[ Z_{\psi\psi} \mathbf{i}_\psi = E_\psi - C_{\psi a}^t \tilde{v}_a \]  \hspace{1cm} (6.22)

The above equations can be stated in compound matrix form as
\[
\begin{bmatrix}
\tilde{Y}_{aa} & -C_{a\psi} \\
C_{\psi a}^t & Z_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_a \\
\mathbf{i}_\psi
\end{bmatrix}
= \begin{bmatrix}
\mathbf{l}_a \\
\mathbf{E}_\psi
\end{bmatrix}
\]

The solution for the unknown nodal voltages \( \mathbf{v}_a \) and the currents \( \mathbf{i}_\psi \) in the removed branches may be obtained by the method used in chapter 5.

**Equations of solution**

From equation (6.21)
\[ \mathbf{v}_a = \tilde{Y}_{aa}^{-1} (\mathbf{l}_a + C_{a\psi} \mathbf{i}_\psi). \]  \hspace{1cm} (6.23)

Substituting for \( \mathbf{v}_a \) into equation (6.22) results in
\[(Z_{\psi\psi} + C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi}) \mathbf{i}_\psi + C_{\psi a}^t \tilde{Y}_{aa}^{-1} \mathbf{l}_a = \mathbf{E}_\psi \]

Let \( \tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} \)
then \( \mathbf{i}_\psi = \tilde{Z}_{\psi\psi}^{-1} (\mathbf{E}_\psi - C_{\psi a}^t \tilde{Y}_{aa}^{-1} \mathbf{l}_a) \)

Substituting for \( \mathbf{i}_\psi \) from the above equation into equation(6.23) the solution for nodal voltages is obtained as
\[
\mathbf{v}_a = (\tilde{Y}_{aa}^{-1} - \tilde{Y}_{aa}^{-1} C_{a\psi} \tilde{Z}_{\psi\psi}^{-1} C_{\psi a} \tilde{Y}_{aa}^{-1}) \mathbf{l}_a + \tilde{Y}_{aa}^{-1} C_{a\psi} \tilde{Z}_{\psi\psi}^{-1} \mathbf{E}_\psi
\]

The practical steps of solution are dealt with in chapter 7 on page 195.

**Representation of asymmetrical coefficient matrix**

Many physical problems are expressed by equations with asymmetrical coefficients. Such equations arise frequently in stress analysis or in electronic circuitry. In power system analysis
such equations can arise, for example, from networks with quad-
rature transformer taps as shown in chapter 2.

It is self-evident that any set of general linear equations can
be represented by an equivalent network. Further, it can be
shown that if the coefficient matrix of the problem is asymmetri-
cal, then the equivalent network is mixed. As an example con-
sider the following equations with asymmetrical coefficients.

\[
\begin{bmatrix}
4 & -1 & 0 \\
-2 & 7 & -4 \\
0 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
\end{bmatrix} =
\begin{bmatrix}
I_a \\
I_b \\
I_c \\
\end{bmatrix}
\]

From the above equations it is possible to construct a mixed net-
work, for example, as shown in fig. 6.4.

![Network Diagram](image)

admittance values shown

\text{Fig. 6.4}

This network is formed by considering first the second equa-
tion, which is

\[-2v_a + 7v_b - 4v_c = I_b\]

which may be written in terms of voltage differences as

\[2(v_b - v_a) + 4(v_b - v_c) + v_b = I_b\]

Therefore, at first, the network is constructed with the branch
admittances \(Y_1 = 2\) mhos between nodes a and b, \(Y_2 = 4\) mhos
between nodes b and c and \(Y_3 = 1\) mho from node b to common
reference.

From the first of the original equations,

\[4v_a - v_b = I_a\]
but between nodes a and b there is a branch with an admittance of 2 mhos, therefore the above equation may be rewritten in terms of voltage differences as

\[ 2(v_a - v_b) - 1(v_a - v_b) + 3v_a = I_a \]

To satisfy this equation the partial network is now extended by adding two parallel branches between node a and the common reference, one with an admittance of \( Y_j = -1 \) mho in series with a voltage source \( e_j = v_b \), and the other admittance of \( Y_4 = 3 \) mhos.

Similarly from the last of the original equations,

\[ 5v_c = I_c \]

But between nodes b and c there is a branch with an admittance of 4 mhos, hence the above equations may be rewritten in terms of voltage difference as

\[ 4(v_c - v_b) - 4(v_c - v_b) + 5v_c = I_c \]

To satisfy this equation the network is completed by adding two parallel branches between node c and the common reference, one with an admittance of \( Y_k = -4 \) mhos in series with a voltage source \( e_k = v_b \) and the other with an admittance \( Y_5 = 5 \) mhos as shown in fig. 6.4.

In general, any set of linear equations with an asymmetrical coefficient matrix can be represented by a mixed network where the voltage sources \( e_{ij} \) are in series with branches connected between the nodes and the common reference and are simply related to the nodal voltages.

For the particular example of fig. 6.4,

\[ e_j = v_b \]

\[ e_k = v_b \]

Expressing in matrix form the relationship between the voltage sources \( e_{ij} \) and the nodal voltages \( v_a \) results in

\[
\begin{bmatrix}
e_j \\
e_k
\end{bmatrix} =
\begin{bmatrix}
. & 1 & . & . & . \\
. & 1 & . & . & .
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f
\end{bmatrix}
\]
or in general terms

$$\mathbf{e}_\psi = \mathbf{A}_\psi \mathbf{v}_a$$

This relationship will be required in the next section.

The element $A_{ja}$ in row j and column a of the connection matrix $\mathbf{A}_\psi$ may be defined as

$$A_{ja} = \begin{cases} 
+1, & \text{when } e_j = v_a \\
-1, & \text{when } e_j = -v_a \\
0, & \text{otherwise}
\end{cases}$$

In general, more than one network can be derived to represent the same set of equations. The complexity of the network representation depends on the manner in which the nodal equations are derived from the matrix of coefficients. The derived networks are all equivalent and, in practice, it is usually convenient to work with the simplest network.

*Equations of solution*

Networks representing equations with an asymmetrical coefficient matrix can be considered as a special case of the general mixed networks described earlier. Because the voltage sources representing the asymmetry are expressed in terms of the unknown nodal voltages, special measures may be taken to simplify the equations of solution for general mixed networks. The simplification may only be obtained if, after dividing, each component network does not contain voltage sources which are a function of nodal voltages belonging to another component network.

Consider the network shown in fig.6.5(a) where all the branches have an impedance of 1 ohm. The unknown nodal voltages $\mathbf{v}_a$ consisting of $v_a$, $v_b$, ..., $v_h$ are all measured with respect to the common reference point. In addition, the unknown voltage sources $e_1$ and $e_m$ are simply related to the unknown nodal voltages as $e_1 = v_d$ and $e_m = v_h$. The current sources $\mathbf{l}_a$ consisting of $l_a$, $l_b$, ..., $l_h$ are assumed to be known.

Removing the two middle branches between nodes d – h and c – f and the two branches containing the voltage sources $e_1 = v_d$ and $e_m = v_h$, the equivalent and removed networks, as shown in fig. 6.5(b) and 6.5(c), can be constructed.

Expressing in matrix form the currents representing the removed branches in terms of all the defined branch currents results in
(a) original network

(b) equivalent network

(c) removed network

Fig. 6.5
or in general terms
\[ \tilde{e}_\psi = \mathbf{C}_{\psi \alpha} \mathbf{v}_\alpha \]  
\hspace{1cm} (6.24)

Expressing in matrix form the hypothetical voltage sources \( \tilde{e}_\psi \) in every mesh of the removed network, in terms of all the defined nodal voltages \( \mathbf{v}_\alpha \), results in
\[
\begin{bmatrix}
\tilde{e}_j \\
\tilde{e}_k \\
\tilde{e}_1 \\
\tilde{e}_m
\end{bmatrix}
= 
\begin{bmatrix}
. & . & -1 & . & . & 1 \\
. & -1 & . & 1 & . & . \\
. & . & . & . & -1 & . \\
. & . & -1 & . & . & .
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f \\
v_g \\
v_h
\end{bmatrix}
\]

or in general terms
\[ \tilde{e}_\psi = -\mathbf{C}^t_{\psi \alpha} \mathbf{v}_\alpha \]  
\hspace{1cm} (6.25)

Expressing in matrix form the relationship between the voltage sources \( e_\psi \), which are in series with the removed branches, and the nodal voltages \( \mathbf{v}_\alpha \) results in
\[
\begin{bmatrix}
e_j \\
e_k \\
e_1 \\
e_m
\end{bmatrix}
\begin{bmatrix}
\ldots \\
\ldots \\
1 \\
\ldots 
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e \\
v_f \\
v_g \\
v_h
\end{bmatrix}
\]

or in matrix form

\[
e_{\psi} = A_{\psi a}^i v_a
\]

(6.26)

The functional relationship for the equivalent network shown in fig. 6.5(b) written in general matrix form is

\[
\tilde{\mathbf{Y}}_{a a} v_a = i_a + \tilde{i}_a
\]

(6.27)

where \( \tilde{\mathbf{Y}}_{a a} \) is a block diagonal matrix and for the particular example consists of two asymmetrical component matrices.

The functional relationship for the removed network, shown in fig. 6.5(c), written in general matrix form is

\[
\mathbf{Z}_{\psi, \psi} i_{\psi} = e_{\psi} + \tilde{e}_{\psi}
\]

(6.28)

For the actual values of the various matrices in equations (6.27) and (6.28) the reader is referred to the numerical example shown on p. 185.

Substituting in equation (6.28) for \( e_{\psi} \) from equation (6.26) and for \( \tilde{e}_{\psi} \) from equation (6.25) results in

\[
\mathbf{Z}_{\psi, \psi} i_{\psi} = A_{\psi a}^i v_a - C_{\psi a}^i v_a
\]

or

\[
\mathbf{Z}_{\psi, \psi} i_{\psi} + B_{\psi a}^t v_a = 0
\]

(6.29)

where

\[
B_{\psi a}^t = C_{\psi a}^t - A_{\psi a}^t
\]

Substituting in equation (6.27) for \( \tilde{i}_a \) from equation (6.24) results in

\[
\tilde{\mathbf{Y}}_{a a} v_a = i_a + C_{\alpha \psi} i_{\psi}
\]

(6.30)
Equations (6.30) and (6.29) can be stated in compound matrix form as

\[
\begin{bmatrix}
\tilde{Y}_{\alpha\alpha} & -C_{\alpha\psi} \\
B_{\psi\alpha}^t & Z_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
v_{\alpha} \\
 i_{\psi}
\end{bmatrix}
= 
\begin{bmatrix}
l_{\alpha} \\
0
\end{bmatrix}
\]

The above equation is of the same form as the fundamental equation of diaoptics developed in chapter 5 with \( B_{\psi\alpha}^t \) in place of \( C_{\psi\alpha}^t \). The equations of solution therefore, can be written by inspection as

\[
\begin{align*}
B_{\psi\alpha}^t &= C_{\psi\alpha}^t - A_{\psi\alpha}^t \\
\tilde{Z}_{\psi\psi} &= Z_{\psi\psi} + B_{\psi\alpha}^t \tilde{Y}_{\alpha\alpha}^{-1} C_{\alpha\psi} \\
i_{\psi} &= -\tilde{Z}_{\psi\psi}^{-1} B_{\psi\alpha}^t \tilde{Y}_{\alpha\alpha}^{-1} l_{\alpha} \\
v_{\alpha} &= Y_{\alpha\alpha}^{-1} (l_{\alpha} + C_{\alpha\psi} i_{\psi})
\end{align*}
\]

or

\[
v_{\alpha} = (\tilde{Y}_{\alpha\alpha}^{-1} - \tilde{Y}_{\alpha\alpha}^{-1} C_{\alpha\psi} \tilde{Z}_{\psi\psi}^{-1} B_{\psi\alpha}^t \tilde{Y}_{\alpha\alpha}^{-1}) l_{\alpha}
\]

The practical steps of solution follow as outlined in chapter 5 with the connection matrix \( B_{\psi\alpha}^t \) in place of \( C_{\psi\alpha}^t \).

**Numerical example**

Consider the network shown in fig. 6.5 where the nodal currents \( l_{\alpha} \) are given as shown below. Constructing the remaining matrices by inspection results in

\[
l_{\alpha} = \begin{bmatrix}
I_a \\
I_b \\
I_c \\
I_d \\
I_e \\
I_f \\
I_g \\
I_h
\end{bmatrix} = \begin{bmatrix}
-12 \\
2 \\
-22 \\
18 \\
2 \\
2 \\
-2 \\
14
\end{bmatrix}
\]
\[
\tilde{\mathbf{Y}}_{aa} = \begin{bmatrix}
3 & -1 & 0 & -1 & . & . & . \\
-2 & 3 & -1 & 0 & . & . & . \\
0 & -1 & 2 & -1 & . & . & . \\
-7 & 0 & -1 & 2 & . & . & . \\
. & . & . & 3 & -1 & -2 & 0 \\
. & . & . & -1 & 2 & 0 & -1 \\
. & . & . & -1 & 0 & 3 & -1 \\
. & . & . & 0 & -1 & -1 & 2 \\
\end{bmatrix}
\]

\[
\mathbf{v}_a = \begin{bmatrix}
\mathbf{v}_a \\
\mathbf{v}_b \\
\mathbf{v}_c \\
\mathbf{v}_d \\
\mathbf{v}_e \\
\mathbf{v}_f \\
\mathbf{v}_g \\
\mathbf{v}_h \\
\end{bmatrix}
\]

\[
\mathbf{C}_{a\psi} = \begin{bmatrix}
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & 1 \\
1 & . & . & . & . & . & . & . & . & . \\
. & . & . & -1 & 1 \\
. & . & . & . & . & . & . & . & . & . \\
-1 & . & . & . & . & . & . & . & . & . \\
\end{bmatrix}
\]

\[
\mathbf{Z}_{\psi\psi} = \begin{bmatrix}
1 & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . \\
\end{bmatrix}
\]

\[
\mathbf{A}^t_{\psi\alpha} = \begin{bmatrix}
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
\end{bmatrix}
\]

\[
\mathbf{B}^t_{\psi\alpha} = \mathbf{C}^t_{\psi\alpha} - \mathbf{A}^t_{\psi\alpha} = \begin{bmatrix}
. & . & . & 1 & . & . & . & -1 \\
. & . & . & 1 & . & . & -1 & . \\
. & . & . & -1 & . & 1 & . & . \\
. & . & 1 & . & . & . & . & -1 \\
\end{bmatrix}
\]
Following the steps shown in chapter 5 on p. 147 with $B^{t}_{\psi,\alpha}$ in place of $C^{t}_{\psi,\alpha}$ results in

(1) $\tilde{Z}_{aa} = \tilde{Y}_{\alpha a}^{-1} =

\begin{bmatrix}
1 & \cdot57143 & \cdot71429 & \cdot85714 & . & . & . & . \\
1 & 1 & 1 & 1 & . & . & . & . \\
1 & \cdot85714 & 1\cdot57143 & 1\cdot28571 & . & . & . & . \\
1 & \cdot71429 & 1\cdot14286 & 1\cdot57143 & . & . & . & . \\
. & . & . & . & 1 & 1 & 1 & 1 \\
. & . & . & . & \cdot85714 & 1\cdot57143 & 1 & 1\cdot28571 \\
. & . & . & . & \cdot57143 & \cdot71429 & 1 & \cdot85714 \\
. & . & . & . & \cdot71429 & 1\cdot14286 & 1 & 1\cdot57143 \\
\end{bmatrix}

(2) $\tilde{Z}_{\psi,\psi} = Z_{\psi,\psi} + B^{t}_{\psi,\alpha} \tilde{Z}_{aa} C_{\alpha,\psi} =

\begin{bmatrix}
-4\cdot14286 & 2\cdot28572 & -1\cdot14286 & 1\cdot14286 \\
2\cdot57142 & 4\cdot14286 & -1\cdot57143 & 2\cdot57143 \\
-2\cdot85714 & -2\cdot71429 & 2\cdot57143 & -1\cdot14286 \\
2\cdot85714 & 2\cdot71429 & 1\cdot14286 & 2\cdot57143 \\
\end{bmatrix}

(3) $Y_{\psi,\psi} = \tilde{Z}_{\psi,\psi}^{-1} =

\begin{bmatrix}
\cdot42857 & -\cdot14286 & \cdot07143 & -\cdot07143 \\
-\cdot03571 & \cdot55357 & \cdot22321 & -\cdot22321 \\
\cdot30357 & \cdot29464 & \cdot70268 & -\cdot00258 \\
-\cdot30357 & -\cdot29464 & -\cdot00268 & \cdot70268 \\
\end{bmatrix}
\[
\begin{align*}
\mathbf{v}'_a &= \mathbf{\tilde{Z}}_{aa} \mathbf{i}_a = \\
&= \begin{bmatrix}
-11.14300 \\
-14.0 \\
-21.71429 \\
-7.42857 \\
\cdots \\
16.0 \\
20.85708 \\
12.57143 \\
23.71428
\end{bmatrix} \\
(4)
\end{align*}
\]

\[
\begin{align*}
\mathbf{v'}_\psi &= \mathbf{B}^\dagger_{\psi\alpha} \mathbf{v}'_\alpha = \\
&= \begin{bmatrix}
31.14286 \\
42.57143 \\
-28.28571 \\
45.42857
\end{bmatrix} \\
\mathbf{i}_\psi &= -\mathbf{Y}^\dagger_{\psi\psi} \mathbf{v'}_\psi = \\
&= \begin{bmatrix}
2 \\
6 \\
2 \\
10
\end{bmatrix} \\
(5) \quad (6)
\end{align*}
\]

\[
\begin{align*}
\mathbf{i}'_\alpha &= \mathbf{C}_{\alpha\psi} \mathbf{i}_\psi = \\
&= \begin{bmatrix}
0 \\
0 \\
16 \\
2 \\
0 \\
-4 \\
0 \\
-2
\end{bmatrix} \\
\mathbf{v}''_a &= \mathbf{\tilde{Z}}_{aa} \mathbf{i}'_\alpha = \\
&= \begin{bmatrix}
13.14286 \\
18.0 \\
27.71428 \\
21.42857 \\
-6.0 \\
-8.85714 \\
-4.57143 \\
-7.71429
\end{bmatrix} \\
(7) \quad (8)
\end{align*}
\]
(9) \( v_a = v'_a + v''_a = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 14 \\ 10 \\ 12 \\ 8 \\ 16 \end{bmatrix} \)

The reader can verify for himself that the solution satisfies the original problem.

Problem
1. By removing the branches shown by broken lines, determine the nodal voltages in the network shown in (a) fig. 6.6, (b) fig. 6.7.
CHAPTER SEVEN

Some Techniques in the Application of Diakoptics

Programming techniques in diakoptical methods
In practice the connection matrix $\mathbf{C}_{\alpha\psi}$ contains a few $+1$ and $-1$ terms, the remainder being 0. Multiplications by such connection matrices to form products $\mathbf{C}_{\alpha\psi} \mathbf{v}_\psi$, $\mathbf{C}_{\psi\alpha}^t \mathbf{v}_\alpha$ or $\mathbf{C}_{\psi\alpha}^t \mathbf{Y}_{\alpha\alpha}^{-1} \mathbf{C}_{\alpha\psi}$ reduce to algebraic summations. The rules for establishing such products by simple summations are shown in the following sections.

Computer storage of connection matrix
For any one removed branch, the corresponding column of the connection matrix $\mathbf{C}_{\alpha\psi}$ will contain $+1$ and $-1$; the rest of the elements will be 0. If the removed branch is connected between a node and reference point, then in the corresponding column only one term other than 0 will appear. For digital computational purposes it is not necessary to store the whole connection matrix; it is sufficient to store only the relevant information in a condensed form.

For example, consider the network shown in fig. 7.1 where the five removed branches with arbitrarily defined currents $i_\psi$ consisting of $i_p, i_q, ..., i_t$ are shown by broken lines. The connection matrix $\mathbf{C}_{\alpha\psi}$ can be established directly by inspection as

$$\mathbf{C}_{\alpha\psi} = \begin{bmatrix}
p & q & r & s & t \\
a & & & & \\
b & & & & \\
c & -1 & & & \\
d & & -1 & & \\
e & & & +1 & \\
f & & +1 & & -1 \\
g & & & & \\
h & & & & -1 \\
i & & +1 & & \\
j & & & & \\
k & & & & +1
d \end{bmatrix}$$

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All the information which is contained in the connection matrix $C_{\alpha\beta}$ can be stored, using less space, as a two-column table as follows:

**TABLE 7.1**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>b</td>
<td>i</td>
<td>k</td>
</tr>
<tr>
<td>q</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For any one removed branch $Z_{\psi} (\psi = p, q, ..., t)$ the branch direction and hence the direction of the current $i_{\psi}$ through the branch is implied by the relative position of node numbers in table 7.1. In this particular case the current flow is in the
direction from the node number shown in the first entry to the
node number shown in the second entry. In power systems analy-
sis these nodes are often referred to as sending-end and receiving-
end respectively. If one end of the removed branch is connected
to the common reference then the corresponding entry is indicated
by 0.

**Product** $C_{a\psi}v_\psi$ or $C_{i\alpha}v_\alpha$

Any multiplication involving the connection matrix $C_{a\psi}$ to form
the product $C_{a\psi}v_\psi$ can be easily expressed as an algebraic sum-
mation using the condensed table 7.1 shown in the previous
section. For example, consider the evaluation of

$$E_\alpha = C_{a\psi}v_\psi$$

where

$$v_\psi^t = [v_p, v_q, v_r, v_s, v_t]$$

This consists of

$$
E_a = 0 \\
E_e = v_p \\
E_i = v_r \\
E_b = -v_r \\
E_f = v_q - v_s \\
E_j = 0 \\
E_c = -v_p \\
E_g = 0 \\
E_k = v_s \\
E_d = -v_q \\
E_h = -v_t
$$

or in general

$$E_\alpha = v_{m_2} - v_{m_1} \quad (\alpha = a, b, c, ... )$$

where $m_1$ = branch number of the row in which $\alpha$
appears in table (7.1), column 1.

$m_2$ = branch number of the row in which $\alpha$
appears in table 7.1 column 2.

If $\alpha$ is not listed in either column the corresponding value of $v$
is zero.

Similarly, consider the evaluation of $E_{\psi}$ from

$$E_{\psi} = C_{i\alpha}v_\alpha$$

where $v_\alpha^t = [v_a, v_b, v_c, v_d, v_e, v_f, v_g, v_h, v_i, v_j, v_k]$

This consists of
\[ E_p = -v_c + v_e \quad E_r = -v_b + v_l \]
\[ E_q = -v_d + v_f \quad E_s = -v_l + v_k \]
\[ E_t = -v_h + 0 \]

or in general terms
\[ E_\psi = v_{n_2} - v_{n_1} \quad (\psi = p, q, r, \ldots) \]

where
\[ n_1 = \text{node number appearing in row } \psi, \text{ column 1 of table 7.1} \]
\[ n_2 = \text{node number appearing in row } \psi, \text{ column 2 of table 7.1.} \]

If \( n_1 \) or \( n_2 \) is not listed in either column the corresponding value of \( v \) is zero.

Using the numbers from table 7.1 as subscripts to identify the variables is particularly convenient for digital computer programming.

**Product** \( C^t_{\psi \alpha} \hat{\mathcal{Y}}^{-1}_{\alpha \alpha} C_{\alpha \psi} \)

The multiplication for the product \( C^t_{\psi \alpha} \hat{\mathcal{Y}}^{-1}_{\alpha \alpha} \) and the multiplication for the product \((C^t_{\psi \alpha} \hat{\mathcal{Y}}^{-1}_{\alpha \alpha})C_{\alpha \psi}\) can be expressed as an algebraic summation; therefore the product \( C^t_{\psi \alpha} \hat{\mathcal{Y}}^{-1}_{\alpha \alpha} C_{\alpha \psi} \) resulting in a square matrix with \( \psi \) rows and columns can also be expressed as a double algebraic summation. The manipulation of matrices which involves multiplication by the connection matrix \( C_{\alpha \psi} \) and its transpose may be reduced to routine steps, particularly convenient for the digital computer, by condensing the connection matrix \( C_{\alpha \psi} \) into a two-column table. For example, the elements of a square matrix \( \hat{Z}_{\psi \psi} \) (\( \psi = p, q, r, s, t \)) where
\[ \hat{Z}_{\psi \psi} = Z_{\psi \psi} + C^t_{\psi \alpha} \hat{\mathcal{Y}}^{-1}_{\alpha \alpha} C_{\alpha \psi} \]
can be constructed from the elements of \( \hat{\mathcal{Y}}^{-1}_{\alpha \alpha} \) and \( Z_{\psi \psi} \) as follows.

**Main diagonal elements**
\[ \hat{Z}_{pp} = Z_p + Z_{cc} + Z_{ee} \]

The diagonal element \( \hat{Z}_{pp} \) of \( \hat{Z}_{\psi \psi} \) in row \( p \) and column \( p \), corresponding to the removed branch \( p \), consists of the algebraic
sum of the branch impedance $Z_p$ and two diagonal elements $Z_{cc}$ and $Z_{ee}$ in row c and column c, and row e and column e, respectively, of $\tilde{\mathbf{Y}}^{-1}_{aa}$. Row c and column e correspond to the node numbers between which the removed branch is connected. If one end of the removed branch is connected to the reference node, only the element of $\tilde{\mathbf{Y}}^{-1}_{aa}$ corresponding to the other end is taken into account.

Non-diagonal elements

$$\tilde{Z}_{pq} = Z_{cd} + Z_{ef}$$

The element $\tilde{Z}_{pq}$ in the row p and column q of $\tilde{Z}_{\psi\psi}$ corresponding to removed branches p and q (connected between nodes c–e and d–f respectively) consists of the algebraic sum of two non-diagonal terms $Z_{cd}$ and $Z_{ef}$ of $\tilde{\mathbf{Y}}^{-1}_{aa}$. If the removed branches p and q are not both terminated in the same component network then the corresponding element of $\tilde{\mathbf{Y}}^{-1}_{aa}$ is zero. If the assumed branch directions of the two removed branches are opposed, that is when the current in one removed branch flows into a given component network and the current in the other removed branch flows away from the same component network, the sign of the corresponding element of $\tilde{\mathbf{Y}}^{-1}_{aa}$ must be changed before adding it to $\tilde{Z}_{\psi\psi}$.

Product $\mathbf{B}_{\psi\alpha}^t \tilde{\mathbf{Y}}^{-1}_{aa} \mathbf{C}_{\alpha\psi}$

The diakoptical solution of networks which include voltage sources representing non-bilateral elements in the removed branches, as shown in chapter 6, involves the evaluation of $\tilde{Z}_{\psi\psi} = Z_{\psi\psi} + B_{\psi\alpha}^t \tilde{\mathbf{Y}}^{-1}_{aa} C_{\alpha\psi}$. The matrix $\tilde{Z}_{\psi\psi}$ can be constructed by inspection from $\tilde{\mathbf{Y}}^{-1}_{aa}$ and $Z_{\psi\psi}$ in a similar manner to that shown in the last section with the following additional qualifications.

Main diagonal element

$$\tilde{Z}_{pp} = Z_p + Z_{cc} + Z_{ee}$$

The presence of a voltage source in the removed branch p does not change the value of the diagonal element $\tilde{Z}_{pp}$. 

Non-diagonal element

\[ \tilde{Z}_{pq} = Z_{cd} + Z_{ef} \]

If neither branch \( p \) or \( q \) contains a voltage source then the element \( \tilde{Z}_{pq} \) does not change. If branch \( p \) is connected between nodes \( c - e \) and branch \( q \) with a voltage source \( v_f \) is connected between node \( d \) and the reference, then

\[ \tilde{Z}_{pq} = Z_{cd} \]
\[ \tilde{Z}_{qp} = Z_{dc} + Z_{fe} \]

where both nodes \( f \) and \( e \) must be in the same component network. If the assumed branch directions do not coincide, that is the currents in the two removed branches are opposed, then the sign of the corresponding element of \( Y^{-1} \) must be changed as before.

**Numerical examples**

![Diagram](image)

Fig. 7.2

For the first example consider the network shown in fig. 7.2. The
nodal impedance matrix $\tilde{Z}_{aa} (= \tilde{Y}_{\alpha\alpha}^{-1})$ of the equivalent network after removing the branches shown by the broken lines is

$$
\tilde{Y}_{\alpha\alpha} = \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ac} & Z_{ad} \\
Z_{ba} & Z_{bb} & Z_{bc} & Z_{bd} \\
Z_{ca} & Z_{cb} & Z_{cc} & Z_{cd} \\
Z_{da} & Z_{db} & Z_{dc} & Z_{dd}
\end{bmatrix}
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
0 & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
Z_{ee} & Z_{ef} & Z_{eg} & Z_{eh} \\
Z_{fe} & Z_{ff} & Z_{fg} & Z_{fh} \\
Z_{ge} & Z_{gf} & Z_{gg} & Z_{gh} \\
Z_{he} & Z_{hf} & Z_{hg} & Z_{hh}
\end{bmatrix}
\begin{bmatrix}
Z_{ii} & Z_{ij} & Z_{ik} & Z_{il} \\
Z_{ji} & Z_{jj} & Z_{jk} & Z_{jl} \\
Z_{ki} & Z_{kj} & Z_{kk} & Z_{kl} \\
Z_{li} & Z_{lj} & Z_{lk} & Z_{ll}
\end{bmatrix}

$$

and the impedance matrix $Z_{\psi\psi}$ of the removed branches is

$$
Z_{\psi\psi} = \begin{bmatrix}
Z_p & \ldots & \ldots & \ldots \\
\ldots & Z_q & \ldots & \ldots \\
\ldots & \ldots & Z_r & \ldots \\
\ldots & \ldots & \ldots & Z_t
\end{bmatrix}
$$

Using the steps outlined in the previous section, the matrix $\tilde{Z}_{\psi\psi} = Z_{\psi\psi} + C_{\psi\alpha}^T \tilde{Y}_{\alpha\alpha}^{-1} C_{\psi\alpha}$ can be established by inspection as

$$
\tilde{Z}_{\psi\psi} = \begin{bmatrix}
(Z_p + Z_{cc} + Z_{ij}) & (Z_{cd} + Z_{ji}) & Z_{ca} & Z_{ji} & Z_{jl} \\
(Z_{dc} + Z_{ij}) & (Z_{q} + Z_{dd} + Z_{ii}) & Z_{da} & Z_{ii} & Z_{il} \\
Z_{ac} & Z_{ad} & (Z_r + Z_{aa} + Z_{ff}) & -Z_{fg} & -Z_{fh} \\
Z_{ij} & Z_{ii} & -Z_{gf} & (Z_s + Z_{ee} + Z_{ll}) & (Z_{gh} + Z_{ii}) \\
Z_{ij} & Z_{il} & -Z_{hf} & (Z_{ng} + Z_{ll}) & (Z_t + Z_{hh} + Z_{ll})
\end{bmatrix}
$$
For the second example consider the network shown in fig. 7.3. The nodal impedance matrix \( Z_{\alpha\alpha} = (\tilde{\gamma}_{\alpha\alpha}^{-1}) \) of the equivalent network after removing the branches shown by broken lines is

\[
\tilde{\gamma}_{\alpha\alpha}^{-1} = \begin{bmatrix}
Z_{aa} & Z_{ab} & Z_{ac} & \cdots & 0 \\
Z_{ba} & Z_{bb} & Z_{bc} & \cdots & 0 \\
Z_{ca} & Z_{cb} & Z_{cc} & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & \ddots & Z_{dd} & Z_{de} & Z_{df} \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
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0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
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0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
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0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddents {2} \phi_{\alpha a} \gamma_{\alpha\alpha}^{-1} C_{\alpha\psi} \text{ can be established by inspection as}

\[
Z_{\psi\psi} \equiv \begin{bmatrix}
Z_p & \cdots & \cdots \\
\cdots & Z_q & \cdots \\
\cdots & \cdots & Z_r \\
\cdots & \cdots & \cdots & Z_s
\end{bmatrix}
\]

Using the steps outlined in the previous section, the matrix

\[
\tilde{Z}_{\psi\psi} = Z_{\psi\psi} + B_{\psi a} \gamma_{\alpha\alpha}^{-1} C_{\alpha\psi}
\]
\[ \tilde{Z}_{\psi\psi} = \begin{bmatrix}
(Z_p + Z_{bb} + Z_{ee}) & (Z_{bc} + Z_{et}) & Z_{be} & -Z_{et} \\
(Z_{cb} + Z_{fe}) & (Z_{a} + Z_{cc} + Z_{tt}) & Z_{cc} & -Z_{tt} \\
(Z_{cb} + Z_{ee}) & (Z_{cc} + Z_{et}) & (Z_{r} + Z_{cc}) & -Z_{et} \\
(-Z_{fe} - Z_{cb}) & (-Z_{tt} - Z_{be}) & -Z_{fc} & (Z_{a} + Z_{tt})
\end{bmatrix} \]

The matrix \( \tilde{Y}_{\alpha\alpha}^{-1} \) may be symmetrical or not but the matrix \( \tilde{Z}_{\psi\psi} \) is always asymmetrical as can be seen from the above example.

**Direct assembly of nodal impedance matrix**

The fundamental equations of diakoptics as developed in chapter 5 can be used to construct a nodal impedance matrix (the inverse of a nodal admittance matrix) from branch data [21]. The method consists of adding one branch at a time to an already formed nodal impedance matrix.

**Addition of a loop branch**

Consider the network shown in fig. 7.4(a) where the full lines indicate a partially formed network for which the nodal impedance matrix \( \tilde{Y}_{\alpha\alpha}^{-1} \) is known. The broken lines indicate a new branch \( p \) which is to be added to the existing network. The new line is connected between two nodes already included in the matrix. The subscript \( \alpha \) indicates the independent nodes, the number of which remain unchanged.

From fig. 7.4 the following relationships can be established

\[
\begin{bmatrix}
\tilde{\gamma}_a \\
\tilde{\gamma}_b \\
\tilde{\gamma}_c
\end{bmatrix} =
\begin{bmatrix}
1 \\
-1 \\
.
\end{bmatrix}
\begin{bmatrix}
i_p
\end{bmatrix}
\]

or in general form as

\[
\tilde{i}_a = \mathbf{C}_{\alpha\psi} i_{\psi}
\]

also

\[
\begin{bmatrix}
\tilde{e}_p \\
v_a \\
v_b \\
v_c
\end{bmatrix} =
\begin{bmatrix}
-1 & 1 & .
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\]
or in general form as

\[ \tilde{e}_{\psi} = -C_{\psi/\alpha}^t v_\alpha \]

The equations describing the equivalent and removed networks are

\[ \tilde{\mathbf{Y}}_{aa} v_\alpha = I_a + \tilde{I}_a \]
\[ Z_{\psi/\psi} I_\psi = \tilde{e}_{\psi} \]

where \( Z_{\psi/\psi} \) is a matrix of order one whose element is the impedance of the new branch, and \( \tilde{\mathbf{Y}}_{aa} \) is the admittance matrix of the
partially formed network without the new branch.

Substituting in the above equations for \( i_\alpha \) and \( \tilde{e}_\psi \) results in

\[
\begin{align*}
\tilde{Y}_{aa} v_\alpha &= l_\alpha + C_{a\psi} i_\psi \\
Z_{\psi\psi} i_\psi &= -C_{\psi a}^t v_\alpha
\end{align*}
\]

Solving the first equation for \( v_\alpha \) and substituting in the second results in

\[
\begin{align*}
v_\alpha &= \tilde{Y}_{aa}^{-1} l_\alpha + \tilde{Y}_{aa}^{-1} C_{a\psi} i_\psi \\
Z_{\psi\psi} i_\psi &= -C_{\psi a}^t \tilde{Y}_{aa}^{-1} l_\alpha - C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} i_\psi
\end{align*}
\]

where \( \tilde{Y}_{aa}^{-1} \) is the known nodal impedance matrix of the partially formed network without the new branch.

Combining the last two equations in a compound matrix form results in

\[
\begin{align*}
\alpha \text{ equations} & \begin{bmatrix} \tilde{Y}_{aa}^{-1} & (\tilde{Y}_{aa}^{-1} C_{a\psi}) \\ (C_{\psi a}^t \tilde{Y}_{aa}^{-1}) & (C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} + Z_{\psi\psi}) \end{bmatrix} \begin{bmatrix} l_\alpha \\ i_\psi \end{bmatrix} = \begin{bmatrix} v_\alpha \\ 0 \end{bmatrix} \\
1 \text{ equation} & \begin{bmatrix} \tilde{Y}_{aa}^{-1} & (\tilde{Y}_{aa}^{-1} C_{a\psi}) \\ (C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} + Z_{\psi\psi}) & 0 \end{bmatrix} \begin{bmatrix} l_\alpha \\ i_\psi \end{bmatrix} = \begin{bmatrix} v_\alpha \\ 0 \end{bmatrix}
\end{align*}
\]

The nodal impedance matrix of the network including the new branch can be obtained by eliminating \( i_\psi \).

The above compound matrix equation can be written in a simplified manner as

\[
\begin{bmatrix}
\tilde{Z}_{aa} & \tilde{Z}_{a\psi} \\
\tilde{Z}_{\psi a} & \tilde{Z}_{\psi\psi}
\end{bmatrix}
\begin{bmatrix} l_\alpha \\ i_\psi \end{bmatrix} = \begin{bmatrix} v_\alpha \\ 0 \end{bmatrix}
\]

where

\[
\begin{align*}
\tilde{Z}_{aa} &= \tilde{Y}_{aa}^{-1} \\
\tilde{Z}_{a\psi} &= \tilde{Y}_{aa}^{-1} C_{a\psi} \\
\tilde{Z}_{\psi a} &= C_{\psi a}^t \tilde{Y}_{aa}^{-1} \\
\tilde{Z}_{\psi\psi} &= C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} + Z_{\psi\psi}
\end{align*}
\]

(a column vector)

(a row vector)

(a scalar).

Multiplying out,

\[
\begin{align*}
\tilde{Z}_{aa} l_\alpha + \tilde{Z}_{a\psi} i_\psi &= v_\alpha \\
\tilde{Z}_{\psi a} l_\alpha + \tilde{Z}_{\psi\psi} i_\psi &= 0
\end{align*}
\]

From the second equation

\[
i_\psi = -\tilde{Z}_{\psi\psi}^{-1} \tilde{Z}_{\psi a} l_\alpha
\]
Substituting for \( i_\psi \) into the first equation

\[
(\tilde{Z}_{aa} - \tilde{Z}_{a\psi} \tilde{Z}_{\psi \psi}^{-1} \tilde{Z}_{\psi a}) i_a = v_a
\]

Therefore the new nodal impedance matrix \( \tilde{Z}_{aa(\text{new})} \) including the additional loop branch is

\[
\tilde{Z}_{aa(\text{new})} = \tilde{Z}_{aa(\text{old})} - \tilde{Z}_{a\psi} \tilde{Z}_{\psi \psi}^{-1} \tilde{Z}_{\psi a}
\]

Because \( C_{a\psi} \) is a vector with one element equal to +1 and another to -1, and the rest 0, then \( \tilde{Z}_{a\psi} \) and \( \tilde{Z}_{\psi a} \) are also vectors and \( \tilde{Z}_{\psi \psi} \) is a scalar. The elements of \( \tilde{Z}_{a\psi}, \tilde{Z}_{\psi a} \) and \( \tilde{Z}_{\psi \psi} \) can be constructed by inspection as shown on p. 206.

**Addition of a radial branch**

Consider the network shown in fig. 7.5(a) where the full lines indicate a partially formed network for which the nodal impedance matrix \( \tilde{Y}_{aa}^{-1} \) is known. The broken lines indicate a new branch \( p \) which is to be added to the existing network. One end of the new branch is connected to a node already included in the matrix \( \tilde{Y}_{aa}^{-1} \) and the other end forms a new node.

The subscript \( \alpha \) indicates the total number of independent nodes of the existing network (three for the particular example) and \( \psi \) indicates the new branches. Since each new branch forms a new node, \( \psi \) also indicates the total number of new nodes.

From fig. 7.5 the following relationships can be established

\[
\begin{bmatrix}
\tilde{i}_a \\
\tilde{i}_b \\
\tilde{i}_c
\end{bmatrix} =
\begin{bmatrix}
1 & & \\
& 1 & \\
& & 1
\end{bmatrix}
\begin{bmatrix}
i_p
\end{bmatrix}
\]

or in general form

\[
\tilde{i}_\alpha = C_{a\psi} i_\psi
\]

Also,

\[
\begin{bmatrix}
e_p
\end{bmatrix} =
\begin{bmatrix}
v_d
\end{bmatrix} -
\begin{bmatrix}
1 & & \\
& 1 & \\
& & 1
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
\]

or in general form

\[
e_\psi = v_\psi - C_{\psi \alpha}^t v_\alpha.
\]
The equations describing the equivalent and removed networks are

\[ \tilde{\mathbf{Y}}_{aa} \mathbf{v}_a = \mathbf{l}_a + \tilde{\mathbf{i}}_a \]

\[ Z_{\psi,\psi} \mathbf{i}_\psi = \tilde{\mathbf{e}}_\psi \]

where \( Z_{\psi,\psi} \) is a matrix of order one whose element is the impedance of the new branch, and \( \tilde{\mathbf{Y}}_{aa} \) is the admittance matrix of the partially formed network without the new branch.

Substituting in the above equations for \( \tilde{\mathbf{i}}_a \) and \( \tilde{\mathbf{e}}_\psi \) results in
\[
\begin{align*}
\tilde{Y}_{aa} v_a &= I_a + C_{a\psi} i_\psi \\
Z_{\psi\psi} i_\psi &= v_\psi - C_{\psi a}^t v_a
\end{align*}
\]

Solving the first equation for \( v_a \) and substituting in the second results in

\[
v_a = \tilde{Y}_{aa}^{-1} I_a + \tilde{Y}_{aa}^{-1} C_{a\psi} i_\psi
\]

\[
Z_{\psi\psi} i_\psi = v_\psi - C_{\psi a}^t \tilde{Y}_{aa}^{-1} I_a - C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} i_\psi
\]

where \( \tilde{Y}_{aa}^{-1} \) is the known nodal impedance matrix of the partially formed network without the new branch.

Combining the last two equations in a compound matrix form results in

\[
\begin{bmatrix}
\tilde{Y}_{aa}^{-1} & (\tilde{Y}_{aa}^{-1} C_{a\psi}) \\
(C_{\psi a}^t \tilde{Y}_{aa}^{-1}) & (C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} + Z_{\psi\psi})
\end{bmatrix}
\begin{bmatrix}
I_a \\
i_\psi
\end{bmatrix}
= 
\begin{bmatrix}
v_a \\
v_\psi
\end{bmatrix}
\]

The above compound matrix equation can be written in a more concise manner as

\[
\begin{bmatrix}
\tilde{Z}_{aa} & \tilde{Z}_{a\psi} \\
\tilde{Z}_{\psi a} & \tilde{Z}_{\psi\psi}
\end{bmatrix}
\begin{bmatrix}
I_a \\
i_\psi
\end{bmatrix}
= 
\begin{bmatrix}
v_a \\
v_\psi
\end{bmatrix}
\]

where

\[
\begin{align*}
\tilde{Z}_{aa} &= \tilde{Y}_{aa}^{-1} \\
\tilde{Z}_{a\psi} &= \tilde{Y}_{aa}^{-1} C_{a\psi} \\
\tilde{Z}_{\psi a} &= C_{\psi a}^t \tilde{Y}_{aa}^{-1} \\
\tilde{Z}_{\psi\psi} &= C_{\psi a}^t \tilde{Y}_{aa}^{-1} C_{a\psi} + Z_{\psi\psi}
\end{align*}
\]

(a column vector)

(a row vector)

(a scalar)

The same compound matrix equation can be written in general terms as

\[
\tilde{Z}_{\delta\delta} I_\delta = v_\delta
\]

where \( v_\delta \) represents all the nodal voltages of the new network and consists of \( v_a, v_b, v_c, v_d \). Similarly, \( I_\delta \) represents all the nodal currents of the new network and consists of \( I_a, I_b, I_c, I_d \) (\( i_\psi = i_p = I_d \)). Therefore \( \tilde{Z}_{\delta\delta} \) represents the nodal impedance matrix of the new network including the new branch and \( \delta \) indicates the total number of independent nodes in the new network.

Because \( C_{a\psi} \) is a vector with one element equal to \(+1\), and
the rest 0, then $\tilde{Z}_{a\psi}$ and $\tilde{Z}_{\psi a}$ are also vectors and $\tilde{Z}_{\psi \psi}$ is a scalar. The elements of $\tilde{Z}_{a\psi}$, $\tilde{Z}_{\psi a}$ and $\tilde{Z}_{\psi \psi}$ can be constructed by inspection as is shown on p. 206.

The addition of a branch connected between a node and reference is a particular case of a branch connected between two nodes of a network. The connection matrix $C_{\alpha \psi}$ corresponding to such an addition consists of a single column with one element equal to +1 and all others 0.

**Practical steps**

The basic steps of the method of direct assembly consist of adding one branch at a time to an already formed nodal impedance matrix. During the process four possible types of connections occur:

*Type 1.* The addition of a branch which has one end connected to a common reference node and the other to a node not previously included in the matrix.

*Type 2.* The addition of a branch which has one end connected to a node already included in the matrix $\tilde{Z}_{aa}$ and the other forming a new node.

*Type 3.* The addition of a branch which has one end connected to a common reference point and the other to a node already included in the matrix $\tilde{Z}_{a\alpha}$.

*Type 4.* The addition of a branch connected between two existing nodes.

For networks without asymmetrical couplings the nodal impedance matrix can be evaluated by adding a branch at a time. For every additional radial branch the existing impedance matrix $\tilde{Z}_{aa}$ is extended by one row and column as shown in table 7.2. Networks with asymmetrical mutual couplings present problems requiring special consideration in each case.

In the table 7.2,

\begin{align*}
Z_p &= \text{impedance of the new branch} \\
Z_a &= \text{column of } \tilde{Z}_{a\alpha} \text{corresponding to node } a \\
Z_b &= \text{column of } \tilde{Z}_{a\alpha} \text{corresponding to node } b \\
Z_{aa} &= \text{element in row } a \text{ and column } a \text{ of } \tilde{Z}_{aa}
\end{align*}
### Table 7.2

**Assembly of nodal impedance matrix**

\[
\begin{bmatrix}
\tilde{Z}_{ss}\n
\tilde{Z}_{aa} & \tilde{Z}_{ap} & \tilde{Z}_{ap} \\
\tilde{Z}_{ap} & \tilde{Z}_{pp} & \tilde{Z}_{pp} \\
\tilde{Z}_{ap} & \tilde{Z}_{pp} & \tilde{Z}_{pp}
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_{aa} \\
\tilde{Z}_{ap} \\
\tilde{Z}_{ap}
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_{aa} \\
\tilde{Z}_{ab} \\
\tilde{Z}_{ac}
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_{ba} \\
\tilde{Z}_{bb} \\
\tilde{Z}_{bc}
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}_{ca} \\
\tilde{Z}_{cb} \\
\tilde{Z}_{cc}
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Type</th>
<th>Network</th>
<th>Components of $\tilde{Z}_{ss}$</th>
<th>Final</th>
</tr>
</thead>
</table>
| 1    | ![Network 1](image1) | $\tilde{Z}_{aa} = Z_{(old)}$  
$\tilde{Z}_{ap} = \tilde{Z}_{pp} = 0$  
$\tilde{Z}_{pp} = Z_p$ | $Z_{(new)} = \tilde{Z}_{ss}$ |
| 2    | ![Network 2](image2) | $\tilde{Z}_{aa} = Z_{(old)}$  
$\tilde{Z}_{ap} = \tilde{Z}_{ap}^1 = Z_a$  
$\tilde{Z}_{pp} = Z_{aa} + Z_p$ | $Z_{(new)} = \tilde{Z}_{ss}$ |
| 3    | ![Network 3](image3) | $\tilde{Z}_{aa} = Z_{(old)}$  
$\tilde{Z}_{ap} = \tilde{Z}_{ap}^1 = Z_a$  
$\tilde{Z}_{pp} = Z_{aa} + Z_p$ | $Z_{(new)} = \tilde{Z}_{aa} - \tilde{Z}_{ap} \tilde{Z}_{pp} \tilde{Z}_{ap}$ |
| 4    | ![Network 4](image4) | $\tilde{Z}_{aa} = Z_{(old)}$  
$\tilde{Z}_{ap} = \tilde{Z}_{ap}^1 = Z_a - Z_b$  
$\tilde{Z}_{pp} = Z_{aa} + Z_{bb} - Z_{ab} + Z_p$ | $Z_{(new)} = \tilde{Z}_{aa} - \tilde{Z}_{ap} \tilde{Z}_{pp} \tilde{Z}_{ap}$ |
Some Techniques in the Application of Diakoptics

\[ Z_{bb} = \text{element in row } b \text{ and column } b \text{ of } \tilde{Z}_{aa} \]
\[ Z_{ab} = \text{element in row } a \text{ and column } b \text{ of } \tilde{Z}_{aa} . \]

The assembling of the nodal impedance matrix has the advantage over the orthodox methods of inversion that it can reduce the amount of calculation. This is because the matrix starts from one element and progressively increases to maximum size. In practice the amount of calculation depends on the order in which the branches are added. The calculations will be kept to a minimum if addition of a loop branch takes preference over a radial branch.

Numerical example

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{network.png}
\caption{Fig. 7.6}
\end{figure}
\end{center}

Consider the network shown in Fig. 7.6 where all the branches have impedance of 1 ohm. Adding the branches in alphabetical order, the nodal impedance matrix of the entire network can be built up as follows.

For branch A the following matrix is formed

\[
\begin{bmatrix}
1
\end{bmatrix}
\]

Adding branch B,

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 
\end{bmatrix}
\]
Adding branch C,
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
\vdots & \vdots & \vdots \\
1 & -1 & a
\end{bmatrix}
\]
where \( a = 1 + 1 - 2 \times 0 + 1 = 3 \)

Eliminating the last row and column,
\[
\begin{bmatrix}
\left(1 - \frac{1 \times 1}{3}\right) & \left(0 + \frac{1 \times 1}{3}\right) \\
\left(0 - \frac{1 \times 1}{3}\right) & \left(1 - \frac{1 \times 1}{3}\right)
\end{bmatrix}
= \frac{1}{3}
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

Adding branch D,
\[
\begin{bmatrix}
2 & 1 & 0 \\
\frac{1}{3} & 1 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

Adding branch E,
\[
\begin{bmatrix}
2 & 1 & 0 & : a \\
1 & 2 & 0 & : b \\
0 & 0 & 3 & : c \\
\vdots & \vdots & \vdots & \vdots \\
a & b & c & : d
\end{bmatrix}
\]
where \( a = \frac{1}{3}(1 - 0) = \frac{1}{3}(1) \)
\( b = \frac{1}{3}(2 - 0) = \frac{1}{3}(2) \)
\( c = \frac{1}{3}(0 - 3) = \frac{1}{3}(-3) \)
\( d = \frac{1}{3}(2 + 3 - 2 \times 0 + 3) = \frac{1}{3}(8) \)

Eliminating the last row and column,
\[
\begin{bmatrix}
\left(2 - \frac{1 \times 1}{8}\right) & \left(1 - \frac{1 \times 2}{8}\right) & \left(0 + \frac{1 \times 3}{8}\right) \\
\frac{1}{3} & \left(1 - \frac{2 \times 1}{8}\right) & \left(2 - \frac{2 \times 2}{8}\right) & \left(0 + \frac{2 \times 3}{8}\right) \\
\left(0 + \frac{3 \times 1}{8}\right) & \left(0 + \frac{3 \times 2}{8}\right) & \left(3 - \frac{3 \times 3}{8}\right)
\end{bmatrix}
= \frac{1}{24}
\begin{bmatrix}
15 & 6 & 3 \\
6 & 12 & 6 \\
3 & 6 & 15
\end{bmatrix}
\]

Adding branch F,
\[
\begin{bmatrix}
15 & 6 & 3 \\
6 & 12 & 6 \\
3 & 6 & 15 \\
\vdots & \vdots & \vdots \\
a & b & c \\
\end{bmatrix}
\]

where

\[
a = \frac{1}{24} (15 - 3) = \frac{1}{24} (12)
\]

\[
b = \frac{1}{24} (6 - 6) = \frac{1}{24} (0)
\]

\[
c = \frac{1}{24} (3 - 15) = \frac{1}{24} (-12)
\]

\[
d = \frac{1}{24} (15 + 15 - 2 \times 3 + 24) = \frac{1}{24} (48)
\]

Eliminating the last row and column,

\[
\begin{bmatrix}
\left(15 - \frac{12 \times 12}{48}\right) & \left(6 - \frac{12 \times 0}{48}\right) & \left(3 + \frac{12 \times 12}{48}\right) \\
\left(6 - \frac{0 \times 12}{48}\right) & \left(12 - \frac{0 \times 0}{48}\right) & \left(6 + \frac{0 \times 12}{48}\right) \\
\left(3 + \frac{12 \times 12}{48}\right) & \left(6 + \frac{12 \times 0}{48}\right) & \left(15 - \frac{12 \times 12}{48}\right)
\end{bmatrix}
\]

\[
\frac{1}{24} = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}
\]

Modification of nodal impedance matrix for changes in a network

In many design problems it is desirable to obtain a series of solutions, each one differing from the previous one by some small change in the basic data. For example, in power systems operational problems the network configuration at a given time is fixed, apart from outage of lines for overhaul or repair. In order that security requirements may be met, it is necessary that no lines shall be overloaded when one or more circuits are removed from the basic network.

For this and other similar analyses a number of solutions with small changes in the network are required. Considerable time is saved by the use of a method for modifying the nodal impedance matrix without the need to formulate and re-invert the whole coefficient matrix for each change in the network.

As an example, consider the removal from the original network shown in fig. 7.7(a) of a branch connected between two nodes resulting in a network as shown in fig. 7.7(b). The removal of a branch can be mathematically expressed as the addition of a branch connected between the same two nodes and with the same admittance value but with the sign changed as shown in fig. 7.7(c).
The numerical procedure is exactly the same as for addition of a branch in the direct assembly of the nodal impedance matrix as shown on p. 205.

In general, the admittance of any branch may be changed to a new value by adding a branch with an appropriate value which may be positive or negative. Any number of changes can be performed simultaneously but it is more practicable, when using a digital computer, to perform one change at a time, repeating the basic processes as many times as there are changes.

**Numerical example**

As an example consider the network shown in fig. 7.8(a) where the branch admittance values are shown. The admittance matrix
\[ \tilde{\mathbf{Y}}_{aa} \] of the network before modification is

\[
\tilde{\mathbf{Y}}_{aa} = \frac{1}{2} \begin{bmatrix}
3 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 3
\end{bmatrix}
\]

and the inverse \( \tilde{\mathbf{Z}}_{aa} = \tilde{\mathbf{Y}}_{aa}^{-1} \) is

\[
\tilde{\mathbf{Z}}_{aa} = \frac{1}{15} \begin{bmatrix}
11 & 3 & 1 \\
3 & 9 & 3 \\
1 & 3 & 11
\end{bmatrix}
\]

Assuming \( \tilde{\mathbf{Z}}_{aa} \) is available, the problem is to obtain \( \mathbf{Z}_{(\text{new})} \) from \( \tilde{\mathbf{Z}}_{aa} \) corresponding to the modified network shown in fig. 7.8(b).

The modified network can be constructed from the original network by adding a branch of admittance \( Y = -0.5 \) connected between the existing nodes b and c. Following the steps shown on p.206, a new matrix is constructed by adding one row and column to the existing matrix as follows:

\[
\tilde{\mathbf{Z}} = \frac{1}{15} \begin{bmatrix}
11 & 3 & 1 & a \\
3 & 9 & 3 & b \\
1 & 3 & 11 & c \\
a & b & c & d
\end{bmatrix} = \frac{1}{15} \begin{bmatrix}
11 & 3 & 1 & 2 \\
3 & 9 & 3 & 6 \\
1 & 3 & 11 & -8 \\
2 & 6 & -8 & -16
\end{bmatrix}
\]

where

\[
a = \frac{1}{15} (3 - 1) = \frac{2}{15} \\
c = \frac{1}{15} (3 - 11) = -\frac{8}{15}
\]

\[
b = \frac{1}{15} (9 - 3) = \frac{6}{15} \\
d = \frac{1}{15} (9 + 11 - 2\times3 - 2\times15) = -\frac{16}{15}
\]

Eliminating the last row and column,

\[
\mathbf{Z}_{(\text{new})} = \frac{1}{15} \begin{bmatrix}
\left(11 + \frac{2\times2}{16}\right) & \left(3 + \frac{6\times2}{16}\right) & \left(1 - \frac{8\times2}{16}\right) \\
\left(3 + \frac{2\times6}{10}\right) & \left(9 + \frac{6\times6}{16}\right) & \left(3 - \frac{8\times6}{16}\right) \\
\left(1 - \frac{2\times8}{16}\right) & \left(3 - \frac{6\times8}{16}\right) & \left(11 - \frac{8\times8}{16}\right)
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
3 & 1 & 0 \\
1 & 3 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]
which is the inverse of the modified network shown in fig. 7.8(b).

Problems

1. Construct by inspection the impedance matrix

\[ \mathbf{Z}_{\psi} = \mathbf{Z}_{\psi} + \mathbf{B}^T \mathbf{Y}^{-1} \mathbf{B} \mathbf{C} \]

for the network shown in fig. 7.9.

2. Assemble the nodal impedance matrix of the network shown in fig. 7.10.
3. Using the results of question 2, find the nodal impedance matrix of the network shown in fig. 7.10 with the lines between nodes a–e and b–d open circuited.
CHAPTER EIGHT

Extended Equations of Diakoptics

In general, electrical network problems can be described by a set of mesh current or nodal voltage equations. The choice often depends on the initial number of variables involved in each method and the convenience of formulating the coefficient matrix. In chapter 5 it was shown that certain advantages can be gained if, by removing individual branches, the network is torn into a number of independent parts.

In this chapter the method of diakoptics is extended to a more general case in which not only individual branches but whole interconnected groups of branches are removed, resulting in a number of independent networks. Some of the independent component parts are described by a set of mesh current equations and the remaining parts by a set of nodal voltage equations. By tearing the networks in this general manner it is possible to choose the most convenient method of description for each independent part and thus to reduce the total number of equations involved. In addition to this, the resulting coefficient matrix is in a block diagonal form which is very convenient for solution.

The functional equations will contain both the mesh currents and nodal voltages as unknowns. The equations of solution may be obtained by first solving for the nodal voltages and then for the mesh currents or vice versa. The choice will depend on the requirements of the solution and on the form of the functional equations, for example, the number of block diagonal matrices in the mesh current and nodal voltage equations.

General functional equations

Some practical problems involve groups of nodes with relatively many branches connected to similar groups by networks with relatively many nodes but few branches. In such cases the number of equations required may be reduced by using mesh current analysis for the parts with many nodes and few branches, and nodal voltage analysis for the parts with many branches and few nodes.

By separating the parts of the network described by nodal voltage equations from the parts described by mesh current
equations, further advantage may be gained from the block diagonal form of the resulting coefficient matrix. Any voltage sources in the parts described by nodal voltage equations can easily be converted to equivalent current sources as shown in chapter 2. Transformation of current sources to equivalent voltage sources introduce such complications that it is not worthwhile. The presence of current sources in any part of a network indicates that nodal analysis is appropriate for that part.

Consider the mixed network shown in fig.8.1(a) where the voltage sources \( \mathbf{E}_p \) consisting of \( E_p, E_q, ..., E_t \) and the nodal currents \( I_\alpha \) consisting of \( I_a, I_b, ..., I_h \) are known.

The problem is to evaluate all branch currents. The required branch currents may easily be obtained from a solution of the unknown mesh currents \( i_\beta \) consisting of \( i_p, i_q, ..., i_t \) and unknown nodal voltages \( v_\alpha \) consisting of \( v_a, v_b, v_c, v_d \) measured with respect to node \( j \), and \( v_e, v_f, v_g, v_h \) measured with respect to node \( k \).

In the particular example it is proposed to remove that part of the network shown by broken lines and in this part the nodal voltages have been arbitrarily defined with respect to nodes \( j \) and \( k \) shown shaded on fig.8.1. The nodal voltages could have been equally well defined with respect to any other node in the respective parts. The parts of the network which are removed and described by nodal equations may be represented in the remaining parts, which are described by mesh current equations, by hypothetical voltage sources \( \mathbf{e}_\beta \) consisting of \( \tilde{e}_r, \tilde{e}_s, \tilde{e}_t \) as shown in fig.8.1(b). The hypothetical voltage sources \( \mathbf{e}_\beta \) are of such magnitude and sense that all the mesh currents are the same as the corresponding mesh currents in the original network.

In a complementary manner, current sources \( i_\alpha \) consisting of \( \tilde{i}_a, \tilde{i}_b, ..., \tilde{i}_t \) are introduced in the nodal network shown in fig.8.1 (c). These hypothetical currents are of such magnitude that all the nodal voltages \( v_\alpha \) are the same as the corresponding ones in the original network. The hypothetical current sources in the nodal network can be expressed in terms of the defined mesh currents by inspection of fig.8.1(a) and (c).

\[
\begin{align*}
\tilde{i}_a &= +i_r \\
\tilde{i}_b &= -i_r \\
\tilde{i}_c &= -i_s \\
\tilde{i}_d &= +i_s
\end{align*}
\]
(a) original network

Fig. 8.1
(b) mesh network

\[ Z_{\beta \beta} i_{\beta} = E_{\beta} + \tilde{e}_{\beta} \]

(c) nodal network

\[ Y_{33} \mathbf{v}_3 = I_3 + \tilde{I}_3 \]
\[ Y_{44} \mathbf{v}_4 = I_4 + \tilde{I}_4 \]

\[ Y_{\alpha \alpha} \mathbf{v}_\alpha = I_\alpha + \tilde{I}_\alpha \]

Fig. 8.1

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\[
\begin{align*}
\tilde{i}_e &= +i_t & \tilde{i}_g &= 0 \\
\tilde{i}_f &= -i_t & \tilde{i}_h &= 0
\end{align*}
\]

Expressing in matrix form the hypothetical current sources \( \tilde{i}_\alpha \) consisting of \( \tilde{i}_a, \tilde{i}_b, \ldots, \tilde{i}_t \) at every node, except the chosen reference nodes \( j \) and \( k \), in terms of all the defined mesh currents results in

\[
\begin{bmatrix}
\tilde{i}_a \\
\tilde{i}_b \\
\tilde{i}_c \\
\tilde{i}_d \\
\tilde{i}_e \\
\tilde{i}_f \\
\tilde{i}_g \\
\tilde{i}_h
\end{bmatrix}
= \begin{bmatrix}
.. & 1 & .. & .. \\
.. & -1 & .. & .. \\
.. & .. & -1 & .. \\
.. & .. & 1 & .. \\
.. & .. & .. & 1 \\
.. & .. & .. & -1 \\
.. & .. & .. & .. \\
.. & .. & .. & .. \\
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
i_t
\end{bmatrix}
\]

or in general terms

\[
\tilde{i}_\alpha = C_{\alpha\beta} i_\beta \quad (8.1)
\]

Similarly, it can be seen from fig. 8.1 that the magnitude of the hypothetical voltage sources \( \tilde{e}_\beta \) in the mesh network representing the nodal network are the same as the voltages across the corresponding nodes.

By inspection of fig. 8.1,

\[
\begin{align*}
\tilde{e}_p &= 0 \\
\tilde{e}_q &= 0 \\
\tilde{e}_r &= v_b - v_a \\
\tilde{e}_s &= v_c - v_d \\
\tilde{e}_t &= v_f - v_e
\end{align*}
\]

Expressing in matrix form the hypothetical voltage sources in every mesh in terms of the defined nodal voltages results in
or in general terms
\[ \tilde{e}_\beta = B_{\beta a} v_a \] (8.2)

At any node where sub-division occurs, say at node a involving the current in mesh p, the corresponding row of connection matrix \( C_{\alpha \beta} \) may be constructed by inspection. The element \( C_{ap} \) in row a and column p of the connection matrix \( C_{a\beta} \) may be defined as

\[
C_{ap} = \begin{cases} 
+1, & \text{if the current in mesh p flows towards node a,} \\
-1, & \text{if the current in mesh p flows from node a,} \\
0, & \text{if the current in mesh p does not flow through node a.}
\end{cases}
\]

In a complementary manner the corresponding column of the matrix \( B_{\beta a} \) may also be constructed by inspection. The element in row p and column a of the connection matrix \( B_{\beta a} \) may be defined as

\[
B_{pa} = \begin{cases} 
-1, & \text{if the current in mesh p flows towards node a,} \\
+1, & \text{if the current in mesh p flows from node a,} \\
0, & \text{if the current in mesh p does not flow through node a.}
\end{cases}
\]

It can be seen from the above definitions that the connection matrix \( C_{a\beta} \) of equation (8.1) is simply related to the connection matrix \( B_{\beta a} \) of equation (8.2) as

\[ B_{\beta a} = -C_{\beta a}^t \]
Therefore

\[ \tilde{e}_\beta = -C_{\beta\alpha}^i v_{\alpha} \]  

(8.3)

The mesh current equations which describe the mesh network shown in fig. 8.1(b) are

\[
\begin{bmatrix}
(Z_2 + Z_3 + Z_4) & -Z_4 & -Z_2 \\
-Z_4 & (Z_4 + Z_5 + Z_6) & -Z_6 \\
-Z_2 & -Z_6 & (Z_1 + Z_2 + Z_6)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix}
= \begin{bmatrix}
E_p + 0 \\
E_q + 0 \\
E_r + \tilde{e}_r
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
(Z_7 + Z_9) & -Z_9 \\
-Z_9 & (Z_8 + Z_9)
\end{bmatrix}
\begin{bmatrix}
i_s \\
i_t
\end{bmatrix}
= \begin{bmatrix}
E_s + \tilde{e}_s \\
E_t + \tilde{e}_t
\end{bmatrix}
\]

The above matrix equations may be written in a more concise form as

\[ Z_{11} i_1 = E_1 + \tilde{e}_1 \]

and

\[ Z_{22} i_2 = E_2 + \tilde{e}_2 \]

Putting the two sets of equations in a compound matrix form as

\[
\begin{bmatrix}
Z_{11} & . \\
. & Z_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
= \begin{bmatrix}
E_1 + \tilde{e}_1 \\
E_2 + \tilde{e}_2
\end{bmatrix}
\]

or in general terms as

\[ Z_{\beta\beta} i_{\beta} = E_{\beta} + \tilde{e}_{\beta} \]  

(8.4)

It should be noted that \( Z_{\beta\beta} \) is not the mesh impedance matrix of the original network fig. 8.1(a) but is that of the network shown in fig. 8.1(b).

In a complementary manner the equations for the nodal network may be established from fig. 8.1(c) as

\[
\begin{bmatrix}
Y_a & -Y_{13} & -Y_{10} & -Y_{12} \\
-Y_{13} & Y_b & -Y_{14} & -Y_{16} \\
-Y_{10} & -Y_{14} & Y_c & -Y_{19} \\
-Y_{12} & -Y_{16} & -Y_{19} & Y_d
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d
\end{bmatrix}
= \begin{bmatrix}
l_a + \tilde{i}_a \\
l_b + \tilde{i}_b \\
l_c + \tilde{i}_c \\
l_d + \tilde{i}_d
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
Y_e & -Y_{23} & -Y_{20} & -Y_{22} \\
-Y_{23} & Y_f & -Y_{24} & -Y_{26} \\
-Y_{20} & -Y_{24} & Y_g & -Y_{29} \\
-Y_{22} & -Y_{26} & -Y_{29} & Y_h
\end{bmatrix}
\begin{bmatrix}
v_e \\
v_f \\
v_g \\
v_h
\end{bmatrix}
=
\begin{bmatrix}
I_e + \tilde{i}_e \\
I_f + \tilde{i}_f \\
I_g + 0 \\
I_h + 0
\end{bmatrix}
\]

where
\[
Y_a = (Y_{10} + Y_{11} + Y_{12} + Y_{13}) \quad Y_e = (Y_{20} + Y_{21} + Y_{22} + Y_{23})
\]
\[
Y_b = (Y_{13} + Y_{14} + Y_{15} + Y_{16}) \quad Y_f = (Y_{23} + Y_{24} + Y_{25} + Y_{26})
\]
\[
Y_c = (Y_{10} + Y_{14} + Y_{18} + Y_{19}) \quad Y_g = (Y_{20} + Y_{24} + Y_{28} + Y_{29})
\]
\[
Y_d = (Y_{12} + Y_{16} + Y_{17} + Y_{19}) \quad Y_h = (Y_{22} + Y_{26} + Y_{27} + Y_{29})
\]

The above matrix equations may be written in a more concise form as
\[
Y_{33}v_3 = I_3 + \tilde{i}_3
\]
\[
\text{and} \quad Y_{44}v_4 = I_4 + \tilde{i}_4
\]

Putting the two sets of equations in a compound form as
\[
\begin{bmatrix}
Y_{33} & \cdot \\
\cdot & Y_{44}
\end{bmatrix}
\begin{bmatrix}
v_3 \\
v_4
\end{bmatrix}
=
\begin{bmatrix}
I_3 + \tilde{i}_3 \\
I_4 + \tilde{i}_4
\end{bmatrix}
\]

or in general terms as
\[
Y_{\alpha\alpha}v_\alpha = I_\alpha + \tilde{i}_\alpha
\]

Substituting for \( \tilde{i}_\alpha \) in the above equation from equation (8.1) and for \( \tilde{\gamma}_\beta \) in equation (8.4) from equation (8.3) results in
\[
Y_{\alpha\alpha}v_\alpha = I_\alpha + C_{\alpha\beta}i_\beta
\]  \hfill (8.5)
\[
Z_{\beta\beta}i_\beta = E_\beta - C_{\beta\alpha}v_\alpha
\]  \hfill (8.5)

Equations (8.5) and (8.6) can be stated in compound matrix form as
\[
\begin{bmatrix}
Y_{\alpha\alpha} & -C_{\alpha\beta} \\
C_{\beta\alpha} & Z_{\beta\beta}
\end{bmatrix}
\begin{bmatrix}
v_\alpha \\
i_\beta
\end{bmatrix}
=
\begin{bmatrix}
I_\alpha \\
E_\beta
\end{bmatrix}
\]

\[
\]
The above equation is identical in form to equation (6.19) for a mixed network and can be solved in a similar manner.

The advantage of the above general method is that by suitable sub-division of a given network into nodal and mesh parts the resulting equations may contain fewer variables than would be the case using mesh or nodal analysis only.

Equations of solution

The equations of solution for the unknown mesh currents $i_\beta$ and the nodal voltages $v_\alpha$ may be obtained from equations (8.5) and (8.6) by eliminating $i_\beta$, solving for $v_\alpha$ and then for $i_\beta$; or by eliminating $v_\alpha$ first, solving for $i_\beta$ and then for $v_\alpha$.

The first method is similar to the mesh current analysis as described in chapter 5 and therefore it is referred to here as the mesh current analysis; the second method is similar to the nodal voltage analysis and therefore is referred to here as the nodal voltage analysis.

Mesh current analysis

From equation (8.6)

$$i_\beta = Z_{\beta\beta}^{-1}(E_\beta - C_{\beta\alpha}^t v_\alpha)$$  \hspace{1cm} (8.7a)

Substituting for $i_\beta$ into equation (8.5)

$$(Y_{aa} + C_{\beta\alpha}^t Z_{\beta\beta}^{-1} C_{\beta\alpha}) v_\alpha - C_{\alpha\beta} Z_{\beta\beta}^{-1} E_\beta = I_a$$  \hspace{1cm} (8.8a)

Let

$$\tilde{Y}_{aa} = Y_{aa} + C_{\beta\alpha}^t Z_{\beta\beta}^{-1} C_{\beta\alpha}$$  \hspace{1cm} (8.9a)

then

$$v_\alpha = \tilde{Y}_{aa}^{-1} I_a + \tilde{Y}_{aa}^{-1} C_{\alpha\beta} Z_{\beta\beta}^{-1} E_\beta$$  \hspace{1cm} (8.10a)

The solution for $i_\beta$ may now be obtained from equation (8.7a).

Nodal voltage analysis

From equation (8.5)

$$v_\alpha = Y_{aa}^{-1}(I_a + C_{\alpha\beta} i_\beta)$$  \hspace{1cm} (8.7b)

Substituting for $v_\alpha$ into equation (8.6)

$$(Z_{\beta\beta} + C_{\beta\alpha}^t Y_{aa}^{-1} C_{\alpha\beta}) i_\beta + C_{\beta\alpha}^t Y_{aa}^{-1} I_a = E_\beta$$  \hspace{1cm} (8.8b)

Let

$$\tilde{Z}_{\beta\beta} = Z_{\beta\beta} + C_{\beta\alpha}^t Y_{aa}^{-1} C_{\alpha\beta}$$  \hspace{1cm} (8.9b)

then

$$i_\beta = \tilde{Z}_{\beta\beta}^{-1} E_\beta - \tilde{Z}_{\beta\beta}^{-1} C_{\alpha\beta} Y_{aa}^{-1} I_a$$  \hspace{1cm} (8.10b)

The solution for $v_\alpha$ may now be obtained from equation (8.7b).
Mesh current analysis (cont.)
The process involves the inversion of $Z_{\beta \beta}$ and $\tilde{Y}_{aa}$ where $Z_{\beta \beta}$ is of the block diagonal form. $\tilde{Y}_{aa}$ is not in general sparse, even when $Y_{aa}$ is of block diagonal form. This method is to be preferred if $\beta$ is appreciably larger than $\alpha$.

Nodal voltage analysis (cont.)
The process involves the inversion of $Y_{aa}$ and $\tilde{Z}_{\beta \beta}$ where $Y_{aa}$ is of the block diagonal form. $Z_{\beta \beta}$ is not in general sparse, even when $Z_{\beta \beta}$ is of block diagonal form. This method is to be preferred if $\alpha$ is appreciably larger than $\beta$.

Sometimes practical problems are such that one group of nodes with high branch density is connected to two or more groups of nodes with relatively few branches between the nodes, for example, as shown in fig. 8.2.

Expressing the problem shown in fig. 8.2 in terms of nodal voltage equations only, results in thirty-eight variables, and using mesh current equations only, results in thirty variables. Subdividing as shown by the broken lines, reduces the number of variables to twenty-seven, consisting of two groups with nine variables each and one with six variables. Since $Z_{\beta \beta}$ will be of order eighteen consisting of two block diagonal matrices each of order nine and $Y_{aa}$ of order six, mesh current analysis would be appropriate in this case.

As a second example consider the schematic diagram shown in fig. 8.3 consisting of three groups of nodes with a high density
of branches in each group, interconnected by two networks with relatively few branches between the nodes.

Expressing the problem in fig. 8.3 in terms of nodal voltage equations only, results in forty-four variables, and using mesh current equations only, results in fifty variables. Subdividing, as shown by the broken lines, reduces the number of variables to thirty-two consisting of three groups with six variables each and two of seven variables each.

Since $Z_{\beta\beta}$ will be of order fourteen and $Y_{aa}$ of order eighteen, consisting of three block diagonal matrices each of order six, nodal voltage analysis would be appropriate in this case.

It can be seen from the above two examples that the method of diakoptics as described in this chapter can reduce the total number of variables involved in the initial solution, as well as retaining the coefficient matrix in a block diagonal form to reduce the storage requirement and computation time.

In general the choice of a method of solution depends not only on the relative number of nodes and meshes but is also influenced by the structure of the network configuration and hence by the matrices $Z_{\beta\beta}$ and $Y_{aa}$. For example, if two or more submatrices are identical it is necessary to invert only one of them. The advantage so gained may be sufficient to justify inverting first the block diagonal matrix containing the identical submatrices.

**Steps of solution**

The steps of solution can be summarized as follows.

1. Subdivide the original network into a number of component parts by removing individual branches or parts of interconnected networks.
2. Establish the connection matrix $C_{a\beta}$.
3. Obtain the solution for the unknown mesh currents and nodal voltages from equations (8.9a), (8.10a), and (8.7a) or from equations (8.9b), (8.10b), and (8.7b).
Mesh current analysis

\[ \dot{Y}_{\alpha\alpha} = Y_{\alpha\alpha} + C_{\alpha\beta} Z_{\beta\beta}^{-1} C_{\beta\alpha} \]  
(8.9a)

\[ \dot{v}_{\alpha} = \dot{Y}_{\alpha\alpha}^{-1} (I_{\alpha} + C_{\alpha\beta} Z_{\beta\beta}^{-1} E_{\beta}) \]  
(8.10a)

\[ i_{\beta} = Z_{\beta\beta}^{-1} (E_{\beta} - C_{\beta\alpha} v_{\alpha}) \]  
(8.7a)

Nodal voltage analysis

\[ \ddot{Z}_{\beta\beta} = Z_{\beta\beta} + C_{\beta\alpha} Y_{\alpha\alpha}^{-1} C_{\alpha\beta} \]  
(8.9b)

\[ i_{\beta} = \ddot{Z}_{\beta\beta}^{-1} (E_{\beta} - C_{\beta\alpha} Y_{\alpha\alpha}^{-1} I_{\alpha}) \]  
(8.10b)

\[ v_{\alpha} = Y_{\alpha\alpha}^{-1} (I_{\alpha} + C_{\alpha\beta} i_{\beta}) \]  
(8.7b)

The above equations may be evaluated in the following steps.

Mesh current analysis

1. \[ Y_{\beta\beta} = Z_{\beta\beta}^{-1} \]
2. \[ \dot{Y}_{\alpha\alpha} = Y_{\alpha\alpha} + C_{\alpha\beta} Y_{\beta\beta} C_{\beta\alpha} \]
3. \[ Z'_{\alpha\alpha} = \dot{Y}_{\alpha\alpha}^{-1} \]
4. \[ i'_{\beta} = Y_{\beta\beta} E_{\beta} \]
5. \[ I'_{\alpha} = C_{\alpha\beta} i'_{\beta} \]
6. \[ I''_{\alpha} = I_{\alpha} + I'_{\alpha} \]
7. \[ v_{\alpha} = Z'_{\alpha\alpha} I''_{\alpha} \]
8. \[ E'_{\beta} = C_{\beta\alpha} v_{\alpha} \]
9. \[ i''_{\beta} = Y_{\beta\beta} E'_{\beta} \]
10. \[ i_{\beta} = i'_{\beta} - i''_{\beta} \]

Nodal voltage analysis

1. \[ Z_{\alpha\alpha} = Y_{\alpha\alpha}^{-1} \]
2. \[ \ddot{Z}_{\beta\beta} = Z_{\beta\beta} + C_{\beta\alpha} Z_{\alpha\alpha} C_{\alpha\beta} \]
3. \[ Y'_{\beta\beta} = \ddot{Z}_{\beta\beta}^{-1} \]
4. \[ v'_{\alpha} = Z_{\alpha\alpha} I_{\alpha} \]
5. \[ E'_{\beta} = C_{\beta\alpha} v'_{\alpha} \]
6. \[ E''_{\beta} = E_{\beta} - E'_{\beta} \]
7. \[ i_{\beta} = Y'_{\beta\beta} E''_{\beta} \]
8. \[ i'_{\alpha} = C_{\alpha\beta} i_{\beta} \]
9. \[ v''_{\alpha} = Z_{\alpha\alpha} I'_{\alpha} \]
10. \[ v_{\alpha} = v'_{\alpha} + v''_{\alpha} \]

Numerical examples

As a first example consider the network shown in fig. 8.4(a) where all the branches have an impedance of 1 ohm. Removing the branches shown by broken lines, the equivalent network consisting of two component parts is constructed as shown in fig. 8.4(b). Transforming the voltage sources in the branches shown by broken lines into equivalent current sources, the various matrices of the equivalent and removed networks shown in fig. 8.4(c) can be constructed by inspection as
(a) original network

(b) equivalent network

(c) removed network

Fig. 8.4
\[
Z_{\beta\beta} = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3 \\
\vdots & \vdots & \vdots \\
. & . & 3 \\
. & . & -1 \\
. & . & -1 \\
\end{bmatrix}
\]

\[
i_{\beta} = \begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
i_t \\
i_u \\
\end{bmatrix}
\]

\[
E_{\beta} = \begin{bmatrix}
10 \\
12 \\
8 \\
-7 \\
6 \\
2 \\
\end{bmatrix}
\]

\[
Y_{aa} = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3 \\
\end{bmatrix}
\]

\[
l_{\alpha} = \begin{bmatrix}
42 \\
-26 \\
-36 \\
\end{bmatrix}
\]

\[
v_{\alpha} = \begin{bmatrix}
v_a \\
v_b \\
v_c \\
\end{bmatrix}
\]

\[
C_{\alpha\beta} = \begin{bmatrix}
. & . & . & -1 & . & . \\
1 & . & . & . & . & . \\
-1 & . & . & . & . & . \\
\end{bmatrix}
\]

Following the steps outlined on p. 225 results in

\[
Y_{\beta\beta} = \frac{1}{4} \begin{bmatrix}
2 & 1 & 1 & . & . & . \\
1 & 2 & 1 & . & . & . \\
1 & 1 & 2 & . & . & . \\
. & . & . & 2 & 1 & 1 \\
. & . & . & 1 & 2 & 1 \\
. & . & . & 1 & 1 & 2 \\
\end{bmatrix}
\]
From the above solution the reader can easily obtain the current in and voltage difference across each individual branch and verify for himself that the solution satisfies the original problem.

As a second example consider the network shown in fig. 8.5(a) where all the branches have an impedance of 1 ohm. Removing the branches shown by broken lines and transforming the voltage sources in the remaining branches to equivalent current sources, the equivalent network consisting of two independent parts is constructed as shown in fig. 8.5(b). The various matrices of the
Fig. 8.5

Equivalent and removed networks as shown in fig. 8.5(c) can be constructed by inspection as

\[ \varnothing \]
\( Y_{\alpha\alpha} = \begin{bmatrix} 3 & -1 & -1 & . & . & . \\ -1 & 3 & -1 & . & . & . \\ -1 & -1 & 3 & . & . & . \\ . & . & . & 3 & -1 & -1 \\ . & . & . & -1 & 3 & -1 \\ . & . & . & -1 & -1 & 3 \end{bmatrix} \)

\( I_\alpha = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -52 \\ 0 \\ 62 \end{bmatrix} \quad V_\alpha = \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \\ v_f \end{bmatrix} \)

\( Z_{\beta\beta} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad I_\beta = \begin{bmatrix} i_p \\ i_q \\ i_r \end{bmatrix} \quad E_\beta = \begin{bmatrix} 9 \\ 10 \\ 3 \end{bmatrix} \)

\( C_{\alpha\beta} = \begin{bmatrix} -1 & . & . \\ 1 & . & . \\ . & . & 1 \\ . & . & -1 \\ . & . & . \end{bmatrix} \)

Following the steps outlined on p. 225 results in:
(1) \[
\mathbf{Z}_{aa} = \frac{1}{4} \begin{bmatrix}
2 & 1 & 1 & \cdots & . & . & . \\
1 & 2 & 1 & \cdots & . & . & . \\
1 & 1 & 2 & \cdots & . & . & . \\
. & . & . & \cdots & 2 & 1 & 1 \\
. & . & . & \cdots & 1 & 2 & 1 \\
. & . & . & \cdots & 1 & 1 & 2 
\end{bmatrix}
\]

(2) \[
\tilde{\mathbf{Z}}_{\beta\beta} = \frac{1}{2} \begin{bmatrix}
7 & -2 & 0 \\
-2 & 8 & -2 \\
0 & -2 & 7 
\end{bmatrix}
\]

(3) \[
\mathbf{Y}_{\beta\beta} = \frac{1}{168} \begin{bmatrix}
52 & 14 & 4 \\
14 & 49 & 14 \\
4 & 14 & 52 
\end{bmatrix}
\]

(4) \[
\mathbf{v}_\alpha^' = \frac{1}{2} \begin{bmatrix}
8 \\
8 \\
8 \\
-21 \\
5 \\
36 
\end{bmatrix}
\]

(5) \[
\mathbf{E}_{\beta}^' = \begin{bmatrix}
0 \\
0 \\
-13 
\end{bmatrix}
\]

(6) \[
\mathbf{E}_{\beta}^" = \begin{bmatrix}
-9 \\
-10 \\
-16 
\end{bmatrix}
\]

(7) \[
\mathbf{l}_\beta = \begin{bmatrix}
4 \\
5 \\
6 
\end{bmatrix}
\]

(8) \[
\mathbf{i}_\alpha^' = \begin{bmatrix}
-4 \\
4 \\
0 \\
6 \\
-6 \\
0 
\end{bmatrix}
\]

(9) \[
\mathbf{v}_\alpha^" = \begin{bmatrix}
-2 \\
-4 \\
0 \\
3 \\
-3 \\
0 
\end{bmatrix}
\]

(10) \[
\mathbf{v}_\alpha = \begin{bmatrix}
3 \\
5 \\
4 \\
-9 \\
1 \\
18 
\end{bmatrix}
\]
From the above solution the reader can easily obtain the current in and voltage difference across each individual branch and verify for himself that the above solution satisfies the original problem.

Expressing either problem of the last two examples by a set of mesh current equations requires nine equations with nine unknowns for each problem and the solution involves the inversion of a ninth-order matrix.

The same problems when subdivided as shown also require nine equations in nine unknowns but the solution involves the inversion of only three third-order matrices with the reduction in storage space by a factor of three and reduction in computation time approximately by a factor of nine. For larger and more complex problems the savings are even more pronounced.

Problems

1. By removing the network shown by the broken lines determine all the nodal voltages for the network shown in fig. 8.7.

2. By removing the network shown by the broken lines determine all the mesh currents for the network shown in fig. 8.6.
Answers to Problems

CHAPTER 1

1. (a) \( \Delta_a = 4 \)  \( \Delta_b = -3 \) \( \Delta_c = 0 \) \( \Delta_d = 206 \)

\[
\mathbf{C} = \begin{bmatrix}
38 & 26 & 2 \\
28 & 17 & 3 \\
0 & 41 & -29
\end{bmatrix}
\]

\[
\mathbf{D} = \begin{bmatrix}
-20 & 22 \\
40 & 46
\end{bmatrix}
\]

2. (1) \( x = 4, \ y = -3, \ z = 1 \)  
(2) \( x = 1, \ y = 2, \ z = 3 \)

4.

\[
\begin{bmatrix}
i_0 \\
i_1 \\
i_2
\end{bmatrix} = \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 \\
1 & \lambda & \lambda^2 \\
1 & \lambda^2 & \lambda
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix}
\]

5. \( i_1 = 1.19056 - j1.35966 \)  \( i_2 = 1.10472 - j0.72017 \)

6.

\[
\mathbf{A}^{-1} = \frac{1}{60}
\begin{bmatrix}
13 & -30 & -23 & 44 & -5 \\
-7 & 30 & 77 & -56 & 35 \\
-19 & 30 & 89 & -32 & 35 \\
2 & 0 & -22 & 16 & -10 \\
1 & -30 & -71 & 68 & -5
\end{bmatrix}
\]

CHAPTER 2

1. (a)

\[
\begin{bmatrix}
(Z_4 + Z_5 + Z_6) & -Z_4 & -Z_5 \\
-Z_4 & (Z_1 + Z_3 + Z_4) & -Z_3 \\
-Z_5 & -Z_3 & (Z_2 + Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix} = \begin{bmatrix}
E_p \\
0 \\
0
\end{bmatrix}
\]

234
(b) \[
\begin{bmatrix}
4 & -2 & -1 & 1 \\
-2 & 4 & -1 & -2 \\
-1 & -1 & 4 & 2 \\
1 & -2 & 2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s \\
\end{bmatrix}
= 
\begin{bmatrix}
E_p \\
0 \\
E_r \\
0 \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
6 & 0 \\
0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
\end{bmatrix}
= 
\begin{bmatrix}
E_p \\
E_q \\
\end{bmatrix}
\]

2. (a) (b) \[\text{Fig. 2.21}\]

3. (a) \[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
5 \\
7 \\
3 \\
\end{bmatrix}
= 
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
7 \\
\end{bmatrix}
= 
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 3 & 0 & -1 \\
-1 & 0 & 2 & -1 \\
0 & -1 & -1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
5 \\
4 \\
3 \\
\end{bmatrix}
= 
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
\end{bmatrix}
\]
CHAPTER 3

1. (a) \[ X = 0.23 \] ..... (b) \[ X_1 = 0.07 \] ..... 
\[ Y = 1.21 \] ..... \[ X_2 = 0.01 \] ..... 
\[ Z = 1.57 \] ..... \[ X_3 = 0.09 \] ..... 
\[ \quad X_4 = 0.22 \] ..... 

2. (a) \[ X_1 = -0.02787 \] (b) \[ X_1 = 1.5 \] 
\[ X_2 = 2.14527 \] \[ X_2 = -3.5 \] 
\[ X_3 = -0.54957 \] \[ X_3 = -4.5 \] 
\[ X_4 = 0.50708 \] \[ X_4 = 1.0 \] 
\[ X_5 = 0.41206 \] \[ X_5 = 3.5 \] 

4. (a) \[
\begin{bmatrix}
21 & -9 \\
-9 & 21
\end{bmatrix}
\begin{bmatrix}
\nu_a \\
\nu_b
\end{bmatrix} =
\begin{bmatrix}
3 \\
33
\end{bmatrix}
\begin{bmatrix}
34 & -16 \\
-16 & 19
\end{bmatrix}
\begin{bmatrix}
\nu_a \\
\nu_b
\end{bmatrix} =
\begin{bmatrix}
-46 \\
79
\end{bmatrix}
\]

5. (a) \[ \nu_a = 8 \] (b) \[ \nu_a = 1 \] 
\[ \nu_b = 16 \] \[ \nu_b = 5 \] 
\[ \nu_c = 7 \] \[ \nu_c = 4 \] 
\[ \nu_d = 5 \] \[ \nu_d = 3 \] 
\[ \nu_e = 2 \] 

CHAPTER 4

1. (a) \[
\begin{bmatrix}
(Z_4 + Z_5 + Z_6) & -Z_4 & -Z_5 \\
-Z_4 & (Z_1 + Z_3 + Z_4) & -Z_3 \\
-Z_5 & -Z_3 & (Z_2 + Z_3 + Z_5)
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r
\end{bmatrix} =
\begin{bmatrix}
E_p \\
E_q \\
E_r
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
4 & -2 & -1 & 1 \\
-2 & 4 & -1 & -2 \\
-1 & -1 & 4 & 2 \\
1 & -2 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q \\
i_r \\
i_s
\end{bmatrix} =
\begin{bmatrix}
E_p \\
E_q \\
E_r \\
E_s
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
6 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q
\end{bmatrix}
=
\begin{bmatrix}
E_p \\
-E_q
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
(16 - \lambda - \lambda^2) & -5\lambda^2 \\
-5\lambda & 10
\end{bmatrix}
\begin{bmatrix}
i_p \\
i_q
\end{bmatrix}
=
\begin{bmatrix}
E_p \\
E_q
\end{bmatrix}
\]

2. (a) \[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix}
=
\begin{bmatrix}
5 \\
7 \\
3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
3 & -1 & -1 & . & . \\
-1 & 3 & -1 & . & . \\
. & -1 & 3 & -1 & . \\
-1 & . & -1 & 3 & -1 \\
. & -1 & . & -1 & -3
\end{bmatrix}
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
v_d \\
v_e
\end{bmatrix}
=
\begin{bmatrix}
10 \\
. \\
. \\
. \\
-5
\end{bmatrix}
\]

CHAPTER 5

1. (a) \[i_p = 2\] \[i_q = 1\] \[i_r = 4\] \[i_s = 3\] \[i_t = 5\] \[i_u = 7\]

(b) \[i_p = 2\cdot182\] \[i_q = 1\cdot727\] \[i_r = 4\cdot182\] \[i_s = 2\cdot818\] \[i_t = 4\cdot273\] \[i_u = 6\cdot818\]

(c) \[i_p = 2\cdot407\] \[i_q = 2\cdot231\] \[i_r = 3\cdot813\] \[i_s = 2\cdot989\] \[i_t = 4\cdot560\] \[i_u = 6\cdot791\]

2. (a) \[v_a = 3\] \[v_b = 2\] \[v_c = 6\] \[v_d = 1\] \[v_e = 7\] \[v_f = 4\]

(b) \[v_a = 3\cdot636\] \[v_b = 2\cdot182\] \[v_c = 6\cdot273\] \[v_d = 0\cdot818\] \[v_e = 6\cdot364\] \[v_f = 3\cdot727\]

(c) \[v_a = 3\cdot366\] \[v_b = 2\cdot317\] \[v_c = 6\cdot228\] \[v_d = 1\cdot675\] \[v_e = 5\cdot642\] \[v_f = 3\cdot772\]
CHAPTER 6

1. (a) \( v_a = 5, \ v_b = 3, \ v_c = 1, \ v_d = 6, \ v_e = 4, \ v_f = 1. \)

(b) \( v_a = 3, \ v_b = 1, \ v_c = 4, \ v_d = 2, \ v_e = 5, \ v_f = 3, \)
\( v_g = 7, \ v_h = 6, \ v_i = 4, \ v_j = 1, \ v_k = 3. \)

CHAPTER 7

1. \[
\begin{bmatrix}
13 & 5 & -5 \\
10 & 8 & -5 \\
-9 & -4 & 8 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
8 & 5 & 3 & 4 & 4 \\
5 & 11 & 6 & 7 & 4 \\
3 & 6 & 9 & 6 & 3 \\
4 & 7 & 6 & 11 & 5 \\
4 & 4 & 3 & 5 & 8 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
11 & 7 & 3 & 2 & 1 \\
7 & 14 & 6 & 4 & 2 \\
3 & 6 & 9 & 6 & 3 \\
2 & 4 & 6 & 14 & 7 \\
1 & 2 & 3 & 7 & 11 \\
\end{bmatrix}
\]

CHAPTER 8

1. \( v_a = -11, \ v_b = -11, \ v_c = -9, \ v_d = -5, \ v_e = 1, \ v_f = -1, \)
\( v_g = -5, \ v_h = -5, \ v_i = -3, \ v_j = 1, \ v_k = -3, \ v_l = 0, \)
\( v_m = -1. \)

2. \( i_1 = 8, \ i_m = 4, \ i_n = 6, \ i_o = 1, \ i_p = 6, \ i_q = 5, \ i_r = 7, \)
\( i_s = 9, \ i_t = 3, \ i_u = 5, \ i_v = 8, \ i_w = 1. \)
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